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# TYPICAL ORBITS OF QUADRATIC POLYNOMIALS WITH A NEUTRAL FIXED POINT II: BRJUNO TYPE

by

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**Abstract.** — We describe the topological behavior of typical *orbits* of complex quadratic polynomials  $P_\alpha(z) = e^{2\pi\alpha i}z + z^2$ , with  $\alpha$  of *high return* type. Here we prove that for such Brjuno values of  $\alpha$  the closure of the *critical orbit*, which is the *measure theoretic attractor* of the map, has zero area. Then combining with Part I of this work, we show that the *limit set* of the orbit of a typical point in the *Julia set* is equal to the closure of the critical orbit.

## Introduction

The local, semi-local, and global dynamics of the maps

$$P_\alpha(z) := e^{2\pi\alpha i}z + z^2 : \mathbb{C} \rightarrow \mathbb{C},$$

for irrational values of  $\alpha$ , have been extensively studied through various methods over the last decades. The aim of this work is to describe the topological behavior of the orbit of typical points under these maps. This is a step toward understanding the measurable dynamics of these maps.

The *post-critical* set of  $P_\alpha$  is defined as the closure of the orbit of the critical value;

$$\mathcal{PC}(P_\alpha) := \overline{\bigcup_{j=1}^{\infty} P_\alpha^{\circ j}(-e^{2\pi\alpha i}/2)}.$$

It is well-known [Lyu83] that  $\mathcal{PC}(P_\alpha)$  is the *measure theoretic attractor* of the dynamics of  $P_\alpha$  on its *Julia set*  $J(P_\alpha)$ . That is, the orbit of Lebesgue almost every point in  $J(P_\alpha)$  eventually stays in any given neighborhood of  $\mathcal{PC}(P_\alpha)$ . To understand the long term behavior of typical orbits in  $J(P_\alpha)$ , one needs to understand the structure of the set  $\mathcal{PC}(P_\alpha)$  and the iterates of  $P_\alpha$  near it.

Two different scenarios occur depending on the local dynamics of  $P_\alpha$  at zero. Let  $\alpha := [0; a_1, a_2, a_3, \dots]$  denote the continued fraction expansion of  $\alpha$  with the convergents  $p_n/q_n := [0; a_1, a_2, \dots, a_n]$ . By classical theorems of Siegel and Brjuno [Sie42, Brj71], if the series  $\sum_{n=1}^{\infty} \log q_{n+1}/q_n$  is finite, the map  $P_\alpha$  is *linearizable*

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at zero, i.e. it is locally conformally conjugate to a rotation. When linearizable, the maximal domain of linearization is called the *Siegel disk* of  $P_\alpha$ . The values of  $\alpha$  for which the above sum is convergent are called *Brjuno* numbers. On the other hand, Yoccoz [Yoc95] proved that the convergence of this series is necessary for the linearizability of  $P_\alpha$ . In [Mañ93], Mañé proves that for irrational  $\alpha$  the orbit of the critical point of  $P_\alpha$  is recurrent, and accumulates at the boundary of the Siegel disk (at zero) when  $P_\alpha$  is linearizable (non-linearizable, respectively).

Petersen and Zakeri in [PZ04] have described the topology and geometry of the dynamics of the linearizable maps  $P_\alpha$  for a.e.  $\alpha \in [0, 1]$ . For these values, they show that  $J(P_\alpha)$  has zero area. However, a rather surprising result of Buff and Chéritat [BC08] states that there are parameters  $\alpha$ , both of Brjuno and non-Brjuno type, for which  $J(P_\alpha)$  has positive area. It is conjectured [Ché09] that for generic values of  $\alpha$ ,  $J(P_\alpha)$  has positive area. So far there is not a single example of a quadratic (or a rational) map with a non-linearizable fixed point whose local dynamics is understood.

A major breakthrough in the field by Inou and Shishikura [IS06] has allowed further progress in the study of these maps. It is an essential part of [BC08] and is used in [Shi10] to show that the boundary of these Siegel disks are Jordan curves. Roughly speaking, Inou-Shishikura show that successive *renormalizations* of  $P_\alpha$  (a sophisticated version of successive return maps depicted in Figure 1) are defined on “large enough” domains, and belong to a compact class of maps. This scheme requires the digits in the expansion of  $\alpha$  to be larger than some constant  $N$ ,<sup>(1)</sup> i.e.

$$\alpha \in \text{Irrat}_{\geq N} := \{[0; a_1, a_2, \dots] \in (0, 1) \mid \inf a_i \geq N\}.$$

In [Che10] we started a systematic study of the measurable dynamics of these maps by quantifying the renormalization scheme of Inou-Shishikura as well as estimating the changes of coordinates between consecutive renormalization levels. This was mainly applied to the study of non-linearizable maps. In particular, we showed that for non-Brjuno values of  $\alpha$ ,  $\mathcal{PC}(P_\alpha)$  is non-uniformly porous<sup>(2)</sup> (and hence has zero area). Here, we improve the estimate on the changes of coordinates obtained in Part I to an infinitesimal one in order to prove the following counterpart.

**Theorem A.** — *For every Brjuno  $\alpha \in \text{Irrat}_{\geq N}$ ,  $\mathcal{PC}(P_\alpha)$  has zero area.*

Note that by Mane’s Theorem, the boundary of the Siegel disk is contained in the post critical set, and hence, must have zero area by the above result. It is not known whether there is a Siegel disk of a quadratic polynomial whose boundary has Hausdorff dimension two. The next statement is an immediate corollary of Theorem A.

**Corollary B.** — *For every Brjuno  $\alpha \in \text{Irrat}_{\geq N}$ , almost every point in  $J(P_\alpha)$  is non-recurrent. In particular, there is no finite absolutely continuous invariant measure on  $J(P_\alpha)$ .*

<sup>(1)</sup>However, they conjecture that  $N = 1$ .

<sup>(2)</sup>A set  $E \subseteq \mathbb{C}$  is said to be non-uniformly porous, if there exists a  $\lambda \in (0, 1)$  satisfying the following property. For every  $z \in E$  there exists a sequence of real numbers  $r_n \rightarrow 0$  such that the ball of radius  $r_n$  about  $z$  contains a ball of radius  $\lambda r_n$  disjoint from  $E$ , for every  $n$ .

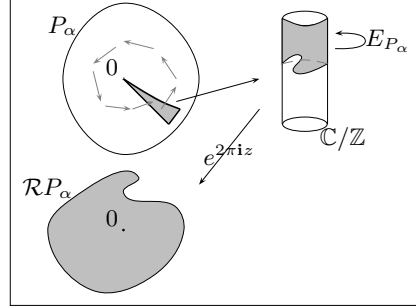


FIGURE 1. Identifying the sides of a sector landing at zero under  $P_\alpha$  one obtains a half infinite cylinder which projects onto a neighborhood of zero under  $e^{2\pi iz}$ . The return map to this sector under  $P_\alpha$  induces a map on that neighborhood,  $\mathcal{R}P_\alpha$ , called the renormalization of  $P_\alpha$ .

Theorem A and its counterpart for non-Brjuno values of  $\alpha$  in Part I enable us to prove the following statement here.

**Theorem C.** — *For all  $\alpha \in \text{Irrat}_{\geq N}$ , the limit set of the orbit of almost every point in  $J(P_\alpha)$  is equal to  $\mathcal{PC}(P_\alpha)$ .*

Let  $\overline{\Delta}_\alpha$  denote the closure of the Siegel disk of  $P_\alpha$ . By [Her85] there are Brjuno values of  $\alpha$  for which  $\overline{\Delta}_\alpha$  does not contain the critical point. By recurrence of the critical point,  $\mathcal{PC}(P_\alpha) \setminus \overline{\Delta}_\alpha$  is non-empty for these values of  $\alpha$ . Conjecturally, this set is homeomorphic to the Cantor bouquet minus its root. Our analysis of the post critical set allows us to prove the following geometric property of these decorations.

**Theorem D.** — *For all  $\alpha \in \text{Irrat}_{\geq N}$ ,  $\mathcal{PC}(P_\alpha)$  is non-uniformly porous at every point in the set  $\mathcal{PC}(P_\alpha) \setminus \overline{\Delta}_\alpha$ .<sup>(3)</sup>*

A new analytical ingredient presented in Section 5 is an infinitesimal estimate on the changes of coordinates (Perturbed Fatou coordinates) between different renormalization levels. That is, given a compact class of maps  $f$  with  $f(0) = 0$  and  $|f'(0)| = 1$ , we give a uniform estimate on the derivative of the perturbed Fatou coordinate with error depending only on the rotation of  $f$  at 0. This is proved by proposing an explicit  $C^2$  change of coordinate that satisfies the optimal estimate and is “nearly conformal and harmonic”. It is an application of Green’s Integral Formula (Hilbert transform) to compare the actual perturbed Fatou coordinate to this change of coordinate. This estimate allows us to prove the results under the sharp Brjuno condition on  $\alpha$ .

The dynamic of  $P_\alpha$  with  $\sup_i a_i < \infty$  has been beautifully described in [McM98]. See [BBCO10, BC04, BC07] and the references therein for some recent advancements in other aspects of the dynamics of these maps.

<sup>(3)</sup>It follows from the proof of this theorem that under various arithmetical conditions on  $\alpha$ , one can replace  $\mathcal{PC}(P_\alpha) \setminus \overline{\Delta}_\alpha$  by  $\mathcal{PC}(P_\alpha)$ . However, we do not know whether one can do this for all Brjuno  $\alpha \in \text{Irrat}_{\geq N}$ .

**Frequently used notations.** —

- $:=$  is used when a notation appears for the first time.
- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$  denote the integer, rational, real, and complex numbers, respectively.
- The bold face  $\mathbf{i}$  denotes the imaginary unit complex number.
- $\operatorname{Re} z$ ,  $\operatorname{Im} z$ , and  $|z|$  denote the real part, the imaginary part, and the absolute value of a complex number  $z$ , respectively.
- $B(y, \delta) \subset \mathbb{C}$  denotes the ball of radius  $\delta$  around  $y$  in the Euclidean metric.
- $\operatorname{int}(S)$  denotes the interior of a set  $S \subset \mathbb{C}$ .
- $f^{\circ n}$  denotes the  $n$  times composition of a map  $f$  with itself,  $f^{\circ 0} = \operatorname{id}$ .
- $\operatorname{Dom} f$ ,  $J(f)$ , and  $\mathcal{PC}(f)$  denote the domain of definition, the Julia set, and the post-critical set of a map  $f$ , respectively.
- Univalent map refers to a one to one holomorphic map.
- Given  $g: \operatorname{Dom} g \rightarrow \mathbb{C}$ , with only one critical point in its domain of definition,  $\operatorname{cp}_g$  and  $\operatorname{cv}_g$  denote the critical point and the critical value of  $g$ , respectively.
- For  $x \in \mathbb{R}$ ,  $\lfloor x \rfloor$  denotes the largest integer less than or equal to  $x$ .

## 1. Preliminaries on renormalization

**1.1. Inou-Shishikura class.** — Consider the cubic polynomial  $P(z) := z(1+z)^2$ . This polynomial has a *parabolic* fixed point at 0, that is, a fixed point of multiplier  $e^{2\pi\alpha\mathbf{i}}$  with  $\alpha \in \mathbb{Q}$ . It has a critical point at  $\operatorname{cp}_P := -1/3$  with  $P(\operatorname{cp}_P) := \operatorname{cv}_P = -4/27$ , and another critical point at  $-1$  which is mapped to 0 under  $P$ .

Consider the ellipse

$$E := \left\{ x + \mathbf{i}y \in \mathbb{C} \mid \left( \frac{x+0.18}{1.24} \right)^2 + \left( \frac{y}{1.04} \right)^2 \leq 1 \right\},$$

and let

$$(1) \quad U := g(\hat{\mathbb{C}} \setminus E), \text{ where } g(z) := -\frac{4z}{(1+z)^2}.$$

The domain  $U$  contains 0 and  $\operatorname{cp}_P$ , but not  $-1$ . Following [IS06], we define the classes of maps

$$\mathcal{IS} := \left\{ f := P \circ \varphi^{-1}: U_f \rightarrow \mathbb{C} \mid \begin{array}{l} \varphi: U \rightarrow U_f \text{ is univalent, } \varphi(0) = 0, \varphi'(0) = 1, \\ \text{and } \varphi \text{ has a quasi-conformal}^{(4)} \text{ extension to } \mathbb{C}. \end{array} \right\},$$

and,

$$\mathcal{IS}_A := \{ f(e^{2\pi\alpha\mathbf{i}}z) \mid f \in \mathcal{IS}, \text{ and } \alpha \in A \}, \text{ where } A \subseteq \mathbb{R}.$$

Abusing the notation,  $\mathcal{IS}_\beta$  denotes the set  $\mathcal{IS}_{\{\beta\}}$ , for  $\beta \in \mathbb{R}$ .

Every map in  $\mathcal{IS}$  has a parabolic fixed point at 0 and a unique critical point at  $\operatorname{cp}_f := \varphi(-1/3) \in U_f$ .

Consider a map  $h: \operatorname{Dom} h \rightarrow \mathbb{C}$ , where  $\operatorname{Dom} h$  denotes the domain of definition (always assumed to be open) of  $h$ . Given a compact set  $K \subset \operatorname{Dom} h$  and an  $\varepsilon > 0$ , a

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<sup>(4)</sup>See [Ahl06] for the definition of quasi-conformal mappings.

neighborhood of  $h$  in the compact-open topology is defined as

$$\mathcal{N}(h; K, \varepsilon) := \{g : \text{Dom } g \rightarrow \mathbb{C} \mid K \subset \text{Dom } g, \text{ and } \sup_{z \in K} |g(z) - h(z)| < \varepsilon\}.$$

In this topology, a sequence  $h_n : \text{Dom } h_n \rightarrow \mathbb{C}$ ,  $n = 1, 2, \dots$ , converges to a map  $h$  if for any neighborhood of  $h$  defined as above,  $h_n$  is contained in that neighborhood for sufficiently large  $n$ . Note that the maps  $h_n$  need not be defined on the same domains.

The class  $\mathcal{IS}_A$  naturally embeds into the space of univalent maps on the unit disk with a neutral fixed point at 0. Therefore, it is a precompact class in the compact-open topology. In particular, it follows from the Kœbe distortion Theorem that  $\{h''(0) \mid h \in \mathcal{IS}_{\mathbb{R}}\}$  is relatively compact in  $\mathbb{C} \setminus \{0\}$ .

Any map  $h = e^{2\pi\alpha\mathbf{i}}f \in \mathcal{IS}_{\alpha}$  has a fixed point at 0 with multiplier  $e^{2\pi\alpha\mathbf{i}}$ . Moreover, if  $\alpha$  is small,  $h$  has another fixed point  $\sigma_h \neq 0$  near 0 in  $U_h$ . The  $\sigma_h$  fixed point depends continuously on  $h$  and has asymptotic expansion  $\sigma_h = -4\pi\alpha\mathbf{i}/f''(0) + o(\alpha)$ , when  $h$  converges to  $f \in \mathcal{IS}$  in a fixed neighborhood of 0. Clearly  $\sigma_h \rightarrow 0$  as  $\alpha \rightarrow 0$ .

The following theorem introduces a useful coordinate to study the local dynamics of maps in  $\mathcal{IS}_{\alpha}$ . See Figure 2 for a geometric description of the following Theorem.

**Theorem 1.1 (Inou–Shishikura [IS06]).** — *There exist positive integers  $\mathbf{k}, \hat{\mathbf{k}}$ , and a real number  $\alpha_* > 0$  such that for every  $h : U_h \rightarrow \mathbb{C}$  in  $\mathcal{IS}_{\alpha}$  (or  $h = P_{\alpha} : \mathbb{C} \rightarrow \mathbb{C}$ ) with  $\alpha \in (0, \alpha_*]$ , there exist a domain  $\mathcal{P}_h \subset U_h$  and a univalent map  $\Phi_h : \mathcal{P}_h \rightarrow \mathbb{C}$  satisfying the following properties:*

- i. *The domain  $\mathcal{P}_h$  is bounded by piecewise smooth curves and is compactly contained in  $U_h$ . Moreover, it contains  $\text{cp}_h$ , 0, and  $\sigma_h$  on its boundary.*
- ii. *There exists a continuous branch of argument defined on  $\mathcal{P}_h$  such that*

$$\max_{w, w' \in \mathcal{P}_h} |\arg(w) - \arg(w')| \leq 2\pi\hat{\mathbf{k}}.$$

- iii.  $\Phi_h(\mathcal{P}_h) = \{w \in \mathbb{C} \mid 0 < \text{Re}(w) < \lfloor 1/\alpha \rfloor - \mathbf{k}\}$ ,  $\text{Im } \Phi_h(z) \rightarrow +\infty$  when  $z \in \mathcal{P}_h \rightarrow 0$ , and  $\text{Im } \Phi_h(z) \rightarrow -\infty$  when  $z \in \mathcal{P}_h \rightarrow \sigma_h$ .
- iv.  $\Phi_h$  satisfies the Abel functional equation on  $\mathcal{P}_h$ , that is,

$$\Phi_h(h(z)) = \Phi_h(z) + 1, \text{ whenever } z \text{ and } h(z) \text{ belong to } \mathcal{P}_h.$$

*Furthermore,  $\Phi_h$  is unique once normalized by  $\Phi_h(\text{cp}_h) = 0$ .*

- v. *The normalized map  $\Phi_h$  depends continuously on  $h$ .*

The map  $\Phi_h : \mathcal{P}_h \rightarrow \mathbb{C}$  obtained in the above theorem is called the *perturbed Fatou coordinate*, or the *Fatou coordinate* for short, of  $h$ .

The class  $\mathcal{IS}$  is denoted by  $\mathcal{F}_1$  in [IS06]. All parts in the above theorem, except the existence of uniform  $\hat{\mathbf{k}}$  and  $\mathbf{k}$  in ii. and iii., follow readily from Theorem 2.1, Main Theorems 1, 3, and Corollary 4.2 in [IS06]. Parts ii. and iii. also follow from those results but require some extra work. A detailed treatment of these statements are given in [BC08, Proposition 12].

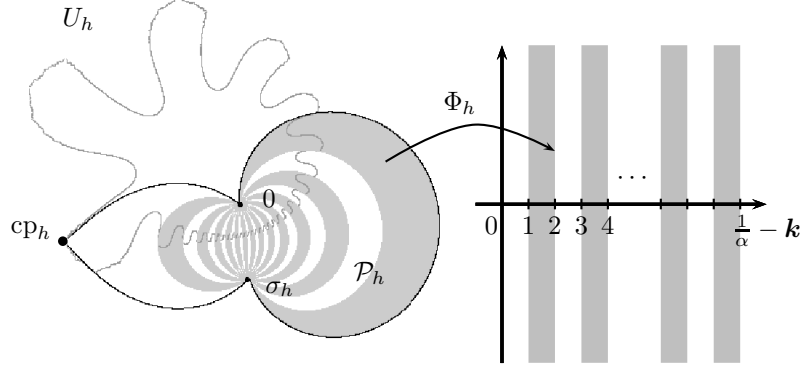


FIGURE 2. A perturbed Fatou coordinate  $\Phi_h$  and its domain of definition  $\mathcal{P}_h$ . The first few iterates of  $cp_h$  under  $h$  appears in the Figure as the boundary of the amoeba containing  $0$ .

**1.2. Renormalization.** — Let  $h: U_h \rightarrow \mathbb{C}$  either be in  $\mathcal{IS}_\alpha$  or be the quadratic polynomial  $P_\alpha$ , with  $\alpha$  in  $(0, \alpha_*]$ . Let  $\Phi_h: \mathcal{P}_h \rightarrow \mathbb{C}$  denote the normalized Fatou coordinate of  $h$ . Define

$$(2) \quad \begin{aligned} \mathcal{C}_h &:= \{z \in \mathcal{P}_h : 1/2 \leq \operatorname{Re}(\Phi_h(z)) \leq 3/2, -2 < \operatorname{Im} \Phi_h(z) \leq 2\}, \text{ and} \\ \mathcal{C}_h^\sharp &:= \{z \in \mathcal{P}_h : 1/2 \leq \operatorname{Re}(\Phi_h(z)) \leq 3/2, 2 \leq \operatorname{Im} \Phi_h(z)\}. \end{aligned}$$

By definition,  $cv_h \in \operatorname{int}(\mathcal{C}_h)$  and  $0 \in \partial(\mathcal{C}_h^\sharp)$ .

Assume for a moment that there exists a positive integer  $k_h$ , depending on  $h$ , with the following properties:

- For every integer  $k$ , with  $0 \leq k \leq k_h$ , there exists a unique connected component of  $h^{-k}(\mathcal{C}_h^\sharp)$  which is compactly contained in  $\operatorname{Dom} h$  and contains  $0$  on its boundary. We denote this component by  $(\mathcal{C}_h^\sharp)^{-k}$ .
- For every integer  $k$ , with  $0 \leq k \leq k_h$ , there exists a unique connected component of  $h^{-k}(\mathcal{C}_h)$  which has non-empty intersection with  $(\mathcal{C}_h^\sharp)^{-k}$ , and is compactly contained in  $\operatorname{Dom} h$ . This component is denoted by  $\mathcal{C}_h^{-k}$ .
- The sets  $\mathcal{C}_h^{-k_h}$  and  $(\mathcal{C}_h^\sharp)^{-k_h}$  are contained in

$$\{z \in \mathcal{P}_h \mid \frac{1}{2} < \operatorname{Re} \Phi_h(z) < \frac{1}{\alpha} - k - \frac{1}{2}\}.$$

- The maps  $h: \mathcal{C}_h^{-k} \rightarrow \mathcal{C}_h^{-k+1}$ , for  $2 \leq k \leq k_h$ , and  $h: (\mathcal{C}_h^\sharp)^{-k} \rightarrow (\mathcal{C}_h^\sharp)^{-k+1}$ , for  $1 \leq k \leq k_h$ , are univalent. The map  $h: \mathcal{C}_h^{-1} \rightarrow \mathcal{C}_h$  is a degree two branched covering.

Let  $k_h$  be the smallest positive integer satisfying the above four properties, and define

$$S_h := \mathcal{C}_h^{-k_h} \cup (\mathcal{C}_h^\sharp)^{-k_h}.$$

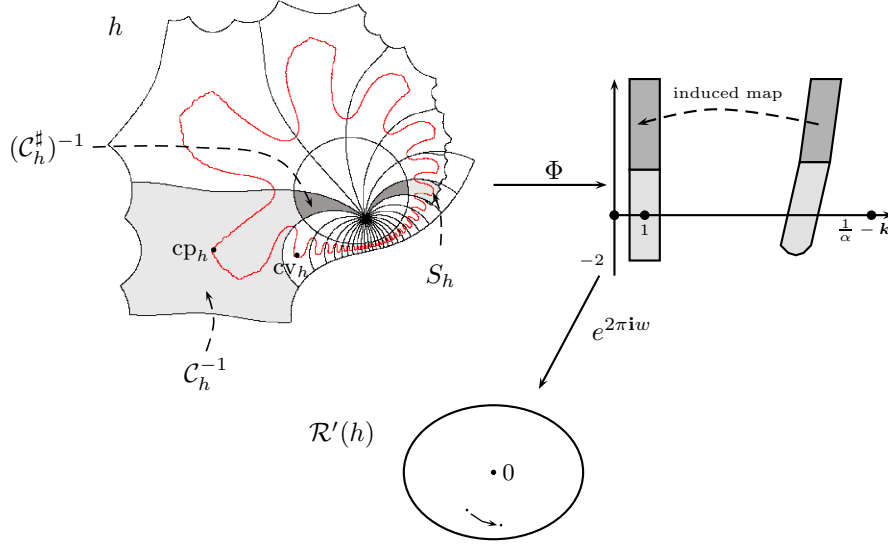


FIGURE 3. The figure shows the sets  $\mathcal{C}_h$ ,  $\mathcal{C}_h^\sharp, \dots, \mathcal{C}_h^{-k_h}$ , and  $(\mathcal{C}_h^\sharp)^{-k_h}$ . The “induced map” projects via  $e^{2\pi i w}$  to a map  $\mathcal{R}(h)$  defined near 0.

Consider the map

$$(3) \quad \Phi_h \circ h^{\circ k_h} \circ \Phi_h^{-1} : \Phi_h(S_h) \rightarrow \mathbb{C}.$$

By the Abel functional equation, this map projects via  $z = \frac{-4}{27}e^{2\pi i w}$  to a well-defined map  $\mathcal{R}'(h)$ , defined on a set containing 0 in its interior. One can see that  $\mathcal{R}(h)$  has asymptotic expansion  $e^{2\pi \frac{-1}{\alpha} i} z + O(z^2)$  near 0, See Figure 3.

The conjugate map  $s \circ \mathcal{R}'(h) \circ s^{-1}$ , where  $s(z) := \bar{z}$  denotes the complex conjugation map, is of the form  $z \mapsto e^{2\pi \frac{1}{\alpha} i} z + O(z^2)$  near 0. The map  $\mathcal{R}(h) := s \circ \mathcal{R}'(h) \circ s^{-1}$ , restricted to the interior of  $s(\frac{-4}{27}e^{2\pi i(\Phi_h(S_h))})$ , is called the *near-parabolic renormalization* of  $h$  by Inou and Shishikura. We simply refer to it as the *renormalization* of  $h$ . It is clear that one time iterating  $\mathcal{R}(h)$  corresponds to several times iterating  $h$ , through the change of coordinates, see Lemma 2.1. For some applications of a closely related renormalization (Douady-Ghys renormalization) one may see [Dou87], [Dou94], [Yoc95], [Shi98] and the references therein.

The following theorem [IS06, Main theorem 3] states that this definition of renormalization  $\mathcal{R}$  can be carried out for perturbations of maps in  $\mathcal{IS}$ . In particular, this

implies the existence of  $k_h$  satisfying the four properties listed in the definition of the renormalization. See [BC08, Proposition 13] for a detailed argument on this, <sup>(5)</sup>.

Define

$$(4) \quad V := P^{-1}\left(B\left(0, \frac{4}{27}e^{4\pi}\right)\right) \setminus ((-\infty, -1] \cup B)$$

where  $B$  is the component of  $P^{-1}(B(0, \frac{4}{27}e^{-4\pi}))$  containing  $-1$  (see Figure 4).

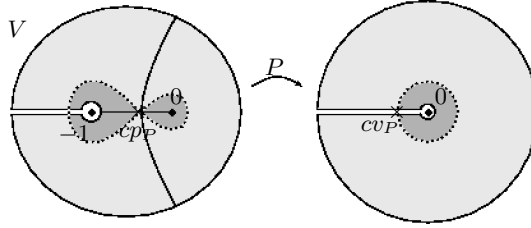


FIGURE 4. A schematic presentation of the polynomial  $P$ ; its domain, and its range. Similar colors and line styles are mapped on one another.

By an explicit calculation (see [IS06, Proposition 5.2]) one can see that the closure of  $U$  is contained in the interior of  $V$ .

**Theorem 1.2 (Inou-Shishikura).** — *There exist a constant  $\alpha^* > 0$  such that if  $h \in \mathcal{IS}_\alpha \cup \{P_\alpha\}$  with  $\alpha \in (0, \alpha^*]$ , then  $\mathcal{R}(h)$  is well-defined and belongs to the class  $\mathcal{IS}_{1/\alpha}$ . Moreover, with the representation  $\mathcal{R}(h) := e^{\frac{2\pi}{\alpha}i} \cdot P \circ \psi^{-1}$ , the map  $\psi : U \rightarrow \mathbb{C}$  extends to a univalent map on  $V$ .*

It follows from a compactness argument that the numbers  $k_h$  are uniformly bounded, see [Che10] for details.

**Lemma 1.3.** — *There exists a  $k'' \in \mathbb{Z}$  such that for every  $h \in \mathcal{IS}_\alpha \cup \{P_\alpha\}$  with  $\alpha \in (0, \alpha^*]$ ,  $k_h \leq k''$ .*

Let  $[0; a_1, a_2, \dots]$  denote the continued fraction expansion of  $\alpha$  as

$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

<sup>(5)</sup>The sets  $\mathcal{C}_h^{-k}$  and  $(\mathcal{C}_h^\sharp)^{-k}$  defined here are (strictly) contained in the sets denoted by  $V^{-k}$  and  $W^{-k}$  in [BC08]. The set  $\Phi_h(\mathcal{C}_h^{-k} \cup (\mathcal{C}_h^\sharp)^{-k})$  is contained in the union

$$D_{-k}^\sharp \cup D_{-k} \cup D_{-k}'' \cup D_{-k+1}' \cup D_{-k+1} \cup D_{-k+1}^\sharp$$

in the notation used in [IS06, Section 5.A].



Define  $\alpha_0 := \alpha$ , and inductively for  $i \geq 1$  define the sequence of real numbers  $\alpha_i \in (0, 1)$  as

$$\alpha_i := \frac{1}{\alpha_{i-1}} \pmod{1}.$$

Then each  $\alpha_i$  has expansion  $[0; a_{i+1}, a_{i+2}, \dots]$ .

If we fix a constant  $N \geq 1/\alpha^*$ , then  $\alpha \in \text{Irrat}_{\geq N}$  implies that  $\alpha_j \in (0, \alpha^*)$ , for  $j = 0, 1, 2, \dots$ . We use this constant  $N$  in the rest of this paper.

Let  $\alpha \in \text{Irrat}_{\geq N}$  and define  $f_0 := P_\alpha$ . Then, using Theorem 1.2, inductively define the sequence of maps

$$f_{n+1} := \mathcal{R}(f_n) : U_{f_{n+1}} \rightarrow \mathbb{C}.$$

Let  $U_n := U_{f_n}$  denote the domain of definition of  $f_n$ , for  $n \geq 0$ . Hence, for every  $n$ ,

$$f_n : U_n \rightarrow \mathbb{C}, \quad f_n(0) = 0, \quad \text{and} \quad f'_n(0) = e^{2\pi\alpha_n i}.$$

## 2. The renormalization tower

### 2.1. Changes of coordinates and sectors around the fixed point. —

**Remark.** — To simplify the technical details of exposition, we assume that

$$(5) \quad N \geq \mathbf{k} + \hat{\mathbf{k}} + 1.$$

The reason for this is to make  $\Phi_{f_n}(\mathcal{P}_{f_n})$  wide enough to contain a set that will be defined in a moment. However, one can avoid this condition by extending  $\Phi_{f_n}$  and  $\Phi_{f_n}^{-1}$  to larger domains, using the dynamics of  $f_n$ .

For  $n \geq 0$ , let  $\Phi_n := \Phi_{f_n}$  denote the Fatou coordinate of  $f_n : U_n \rightarrow \mathbb{C}$  defined on the set  $\mathcal{P}_n := \mathcal{P}_{f_n}$ . For our convenience we define the covering map

$$\mathbb{E}x p(\zeta) := \zeta \mapsto \frac{-4}{27} s(e^{2\pi i \zeta}) : \mathbb{C} \rightarrow \mathbb{C}^*, \quad \text{where } s(z) = \bar{z}.$$

By part ii of Theorem 1.1, Inequality (5), and that  $\mathcal{P}_n$  is simply connected, there is an inverse branch of  $\mathbb{E}x p$  that maps  $\mathcal{P}_n$  into the range of  $\Phi_{n-1}$ ;

$$\eta_n : \mathcal{P}_n \rightarrow \Phi_{n-1}(\mathcal{P}_{n-1}).$$

There may be several choices for this map but we choose one of them (for each  $n$ ) with

$$\text{Re}(\eta_n(\mathcal{P}_n)) \subset [0, \hat{\mathbf{k}} + 1],$$

and fix this choice for the rest of this note. Now, define

$$\psi_n := \Phi_{n-1}^{-1} \circ \eta_n : \mathcal{P}_n \rightarrow \mathcal{P}_{n-1}.$$

Each  $\psi_n$  extends continuously to  $0 \in \partial \mathcal{P}_n$  by mapping it to 0. For  $n \geq 2$  we can form the compositions

$$\Psi_n := \psi_1 \circ \psi_2 \circ \dots \circ \psi_n : \mathcal{P}_n \rightarrow \mathcal{P}_0.$$

For every  $n \geq 0$ , let  $\mathcal{C}_n$  and  $\mathcal{C}_n^\sharp$  denote the corresponding sets for  $f_n$  defined in (2) (i.e., replace  $h$  by  $f_n$ ). Denote by  $k_n$  the smallest positive integer with

$$S_n^0 := \mathcal{C}_n^{-k_n} \cup (\mathcal{C}_n^\sharp)^{-k_n} \subset \{z \in \mathcal{P}_n \mid 0 < \text{Re } \Phi_n(z) < \lfloor \frac{1}{\alpha_n} \rfloor - \mathbf{k} - 1\}.$$

For every  $n \geq 0$  and  $i \geq 2$ , define the sectors

$$S_n^1 := \psi_{n+1}(S_{n+1}^0) \subset \mathcal{P}_n, \text{ and} \\ S_n^i := \psi_{n+1} \circ \cdots \circ \psi_{n+i}(S_{n+i}^0) \subset \mathcal{P}_n.$$

By definition, the critical value of  $f_n$  is contained in  $f_n^{\circ k_n}(S_n^0)$ . Also, all these sectors contain 0 on their boundaries. We will mainly work with  $S_0^i$ , for  $i \geq 0$ .

**Lemma 2.1.** — *Let  $z \in \mathcal{P}_n$  be a point with  $w := \mathbb{E} \exp \circ \Phi_n(z) \in U_{n+1}$ . There exists an integer  $\ell_z$  with  $1 \leq \ell_z \leq \lfloor 1/\alpha_n \rfloor - k + k_n - 1$ , such that*

- *the finite orbit  $z, f_n(z), f_n^{\circ 2}(z), \dots, f_n^{\circ \ell_z}(z)$  is defined, and  $f_n^{\circ \ell_z}(z) \in \mathcal{P}_n$ ;*
- *$\mathbb{E} \exp \circ \Phi_n(f_n^{\circ \ell_z}(z)) = f_{n+1}(w)$ .*

*Proof.* — As  $w \in \text{Dom } f_{n+1}$ , by the definition of renormalization  $\mathcal{R}(f_n) = f_{n+1}$ , there are  $\zeta \in \Phi_n(S_n^0)$  and  $\zeta' \in \Phi_n(\mathcal{C}_n \cup \mathcal{C}_n^\#)$ , such that

$$\mathbb{E} \exp(\zeta) = w, \quad \mathbb{E} \exp(\zeta') = f_{n+1}(w), \text{ and } \zeta' = \Phi_n \circ f_n^{\circ k_n} \circ \Phi_n^{-1}(\zeta).$$

Since  $\mathbb{E} \exp(\Phi_n(z)) = w$ , there exists an integer  $\ell$  with

$$-k_n + 1 \leq \ell \leq \lfloor 1/\alpha_n \rfloor - k - 1,$$

such that  $\Phi_n(z) + \ell = \zeta$ .

By the Abel functional equation for  $\Phi_n$ , we have

$$\begin{aligned} \zeta' &= \Phi_n \circ f_n^{\circ k_n} \circ \Phi_n^{-1}(\zeta) \\ &= \Phi_n \circ f_n^{\circ k_n} \circ \Phi_n^{-1}(\Phi_n(z) + \ell) \\ &= \Phi_n \circ f_n^{\circ k_n + \ell}(z). \end{aligned}$$

Letting  $\ell_z := k_n + \ell$ , we have

$$1 \leq \ell_z \leq k_n + \lfloor 1/\alpha_n \rfloor - k - 1, \quad f_n^{\circ \ell_z}(z) = \Phi_n^{-1}(\zeta') \in \mathcal{P}_n, \text{ and}$$

$$\begin{aligned} \mathbb{E} \exp \circ \Phi_n(f_n^{\circ \ell_z}(z)) &= \mathbb{E} \exp \circ \Phi_n(\Phi_n^{-1}(\zeta')) \\ &= \mathbb{E} \exp(\zeta') \\ &= f_{n+1}(w). \end{aligned}$$

□

It is clear that in the above lemma there are many choices for  $\ell_z$ . In the following lemmas, proved in Part I, by making some specific choices for  $\ell_z$ , and inductively using the above lemma the number of iterates on level  $n$  are related to the number of iterates on 0

Define

$$\mathcal{P}'_n := \{w \in \mathcal{P}_n \mid 0 < \text{Re } \Phi_n(w) < \lfloor 1/\alpha_n \rfloor - k - 1\}.$$

**Lemma 2.2.** — *For every  $n \geq 1$  we have*

- i. *for every  $w \in \mathcal{P}'_n$ ,  $f_0^{\circ q_n} \circ \Psi_n(w) = \Psi_n \circ f_n(w)$ ,*
- ii. *for every  $w \in S_n^0$ ,  $f_0^{\circ (k_n q_n + q_n - 1)} \circ \Psi_n(w) = \Psi_n \circ f_n^{\circ k_n}(w)$ , and*

**2.2. The approximating neighborhoods.** — For every  $n \geq 0$ , consider the union

$$(6) \quad \Omega_n^0 := \bigcup_{i=0}^{k_n + \lfloor 1/\alpha_n \rfloor - \mathbf{k} - 1} f_n^{\circ i}(S_n^0) \subset \text{Dom } f_n.$$

Using Lemma 2.2, we transfer the iterates in the above union to the dynamic plane of  $f_0$  to define

$$\Omega_0^n := \bigcup_{i=0}^{q_n(k_n + \lfloor 1/\alpha_n \rfloor - \mathbf{k} - 1) + q_{n-1}} f_0^{\circ i}(S_0^n) \bigcup \{0\}.$$

The upper bound in the above union is obtained as follows. The first  $k_n$  iterates in (6) correspond to  $k_n q_n + q_{n-1}$  number of iterates on level 0 by Lemma 2.2-ii, and the remaining  $\lfloor 1/\alpha_n \rfloor - \mathbf{k} - 1$  iterates in (6) amounts to  $q_n(\lfloor 1/\alpha_n \rfloor - \mathbf{k} - 1)$  number of iterates by Lemma 2.2-i. The number of iterates to define these unions are chosen so that they satisfy the following property, see Part I for proof.

**Proposition 2.3.** — *For every  $n \geq 0$ , we have*

- i.  $\Omega_0^{n+1}$  is compactly contained in the interior of  $\Omega_0^n$ ;
- ii.  $\mathcal{PC}(f_0)$  is contained in the interior of  $\Omega_0^n$ .

As  $\mathcal{PC}(f_0)$  is the measure theoretic attractor of the map, by Proposition 2.3-ii, for every  $n \geq 0$  and almost every  $z \in J(P_\alpha)$  there is a positive integer  $k$  such that the orbit of  $f^{\circ k}(z)$  is contained in  $\Omega^n$ . Indeed, we prove in Part I that while the orbit of a point stays in some  $\Omega^n$ , it visits all the sectors involved in that union.

Using compactness of the class  $\mathcal{IS}$  and Theorem 1.1-v, we also showed in Part I that each  $\Omega_n^0$  is compactly contained in  $\text{Dom } f_n$ . This statement is uniform in the following sense.

**Lemma 2.4.** — *There exists a positive constant  $\delta$  such that for every  $n \geq 1$  and every  $\xi \in \mathbb{C}$  with  $\mathbb{E}\text{xp}(\xi) \in \Omega_n^0$  we have*

- $\forall j \in \mathbb{Z}, \mathbb{E}\text{xp}(B(j, \delta)) \subset \text{int}(C_n) \subset \Omega_n^0$ .
- $\mathbb{E}\text{xp}(B(\xi, \delta)) \subset \text{Dom } f_n$ ,

### 3. Area of the post-critical set

In this section we prove the following proposition which combined with Proposition 2.3 implies Theorem A. Let  $N$  be the constant introduced at the end of Section 1.

**Proposition 3.1.** — *For all Brjuno  $\alpha$  in  $\text{Irrat}_{\geq N}$ ,  $(\cap_{n=0}^\infty \Omega_0^n) \cap J(P_\alpha)$  has zero area.*

The proof of the above proposition is based on two estimates on the perturbed Fatou coordinates. The first one is summarized in the following lemma and is suited for points in the intersection that are well exposed to the complement of infinitely many  $\Omega_0^n$ .

Given a set  $X \subseteq \mathbb{C}$  and  $\delta > 0$ , define  $B_\delta(X) := \cup_{x \in X} B(x, \delta)$ .

**Lemma 3.2.** — *Given any constant  $E$ , there are positive constants  $\delta_1$ ,  $\delta_2$ , and  $r^*$  such that for every  $n \geq 1$  and every  $\zeta \in \mathbb{C}$  with  $\text{Im } \zeta \leq \frac{1}{2\pi} \log \frac{1}{\alpha_{n+1}} + E$  and  $\mathbb{E}\text{xp}(\zeta) \in \Omega_{n+1}^0$ , there exists a line segment  $\gamma_n : [0, 1] \rightarrow \mathbb{C}$  with  $\gamma_n(0) = \zeta$ , satisfying the following properties:*

- i.  $\mathbb{E}\text{xp} \left( B_{\delta_1} (B(\gamma_n(1), r^*) \cup \gamma_n[0, 1]) \right) \subseteq \text{Dom } f_{n+1} \setminus \{0\}$ ,
- ii.  $\mathbb{E}\text{xp} (B(\gamma_n(1), r^*)) \cap \Omega_{n+1}^0 = \emptyset$ ,  $f_{n+1} (\mathbb{E}\text{xp} (B(\gamma_n(1), r^*))) \cap \Omega_{n+1}^0 = \emptyset$ ,
- iii.  $\text{diameter } \text{Re} \left( B_{\delta_1} (B(\gamma_n(1), r^*) \cup \gamma_n[0, 1]) \right) \leq 1 - \delta_1$ ,
- iv.  $\text{mod } B_{\delta_1} (B(\gamma_n(1), r^*) \cup \gamma_n[0, 1]) \setminus (B(\gamma_n(1), r^*) \cup \gamma_n[0, 1]) \geq \delta_2$ . <sup>(1)</sup>

*Proof.* — This has been proved in Part I when  $E = 0$ . The statement for arbitrary  $E$  immediately follows from the one for  $E = 0$ .  $\square$

The other estimate, stated in Lemma 5.7, is mainly intended for points on the boundary of the Siegel disk.

**3.1. The heights in the tower.** — For every  $n \geq 1$ , let  $\text{Fil}(\Omega_n^0)$  denote the set obtained from adding the bounded components of  $\mathbb{C} \setminus \Omega_n^0$  to  $\Omega_n^0$ , if there is any. For all  $n \geq 1$  and  $j$  with  $0 \leq j < \frac{1}{\alpha_n} - \mathbf{k}$ , let  $I_{n,j}$  denote the connected component of

$$\text{Fil}(\Omega_n^0) \cap \Phi_n^{-1} \{j + ti : t \in \mathbb{R}\},$$

containing zero on its boundary. Each  $I_{n,j}$  is a smooth curve in  $\text{Fil}(\Omega_n^0)$  connecting the boundary of  $\Omega_n^0$  to 0. This implies that there is a continuous inverse branch of  $\mathbb{E}\text{xp}$  defined on every  $\Omega_n^0 \setminus I_{n,j}$ .

By Theorem 1.1-ii, Lemma 1.3, and the precompactness of the class  $\mathcal{IS}_{(0, \alpha^*]}$ , there exists a positive integer  $\mathbf{k}'$  such that

$$\forall n \geq 1 \text{ and } \forall j \text{ with } 0 \leq j < \frac{1}{\alpha_n} - \mathbf{k}, \quad \sup_{z, w \in \Omega_n^0 \setminus I_{n,j}} |\arg(w) - \arg(z)| \leq 2\pi \mathbf{k}',$$

for any continuous branch of argument defined on  $\Omega_n^0 \setminus I_{n,j}$ . To simplify the technical details of the exposition, we assume the following condition on the rotations

$$N \geq 2\mathbf{k}' + \mathbf{k} + \mathbf{k}''.$$

Choose a positive constant

$$\delta_3 \leq \min\{\delta, \delta_1, 1/8\},$$

where  $\delta$  and  $\delta_1$  are obtained in Lemmas 2.4 and 3.2.

To each non-zero  $z_0 \in \cap_{n=0}^\infty \Omega_n^0 \cap J(P_\alpha)$ , we associate a sequence of quadruples

$$(7) \quad \{(z_i, w_i, \zeta_i, \sigma(i))\}_{i=0}^\infty,$$

where,

$$z_i, w_i \in \text{Dom } f_i, \zeta_i \in \Phi_i(\mathcal{P}_i), \text{ and } \sigma(i) \in \mathbb{Z}^+.$$

(This is the same sequence considered in Part I). Define the two sets (see Figure 3.1),

$$\mathcal{A}_n := \{z \in \mathcal{P}_n \mid \mathbf{k}' + 1/2 \leq \text{Re } \Phi_n(z) \leq \lfloor 1/\alpha_n \rfloor - \mathbf{k}, \text{ or } \Phi_n(z) \in \cup_{j=1}^{\mathbf{k}'} B(j, \delta_3)\}$$

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<sup>(1)</sup>mod denotes the conformal modulus of an annulus.

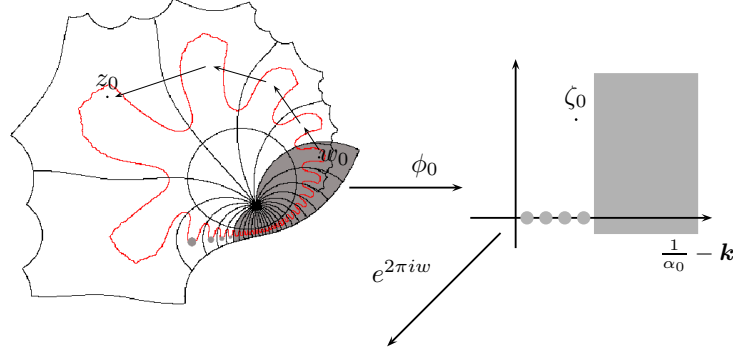


FIGURE 5. The two different colors correspond to the two different ways of going down the renormalization tower. The gray part corresponds to  $\mathcal{A}$  and the rest to  $\mathcal{B}$ .

$$\mathcal{B}_n := \{z \in \mathcal{P}_n \mid 0 \leq \operatorname{Re} \Phi_n(z) < \mathbf{k}' + 1/2 \text{ and } \Phi_n(z) \notin \cup_{j=1}^{\mathbf{k}'} B(j, \delta_3)\} \cup (\Omega_n^0 \setminus \mathcal{P}_n).$$

The sequence (7) is inductively defined as follows. Since  $z_0 \in \Omega_0^0$ , either  $z_0 \in \mathcal{A}_0$  or  $z_0 \in \mathcal{B}_0$ . If  $z_0 \in \mathcal{A}$ , define  $w_0 := z_0$ ,  $\sigma(0) := 0$ , and  $\zeta_0 := \Phi_0(w_0)$ . If  $z_0 \in \mathcal{B}$ , let  $w_0 \in S_0^0$  and positive integer  $\sigma(0) \leq k_0 + \mathbf{k}'$  be such that  $f_0^{\circ \sigma(0)}(w_0) = z_0$ . The point  $w_0$  satisfying this property is not necessarily unique. However, one can choose any of them. The positive integer  $\sigma(0)$  is uniquely determined. Indeed, when  $\sigma(0) \leq k_0 - 1$  or  $|z_0|$  is small enough, such  $w_0$  is unique. Otherwise, there are at most two choices for  $w_0$ . Now, let  $\zeta_0 := \Phi_0(w_0)$ . This finishes the definition of the first quadruple  $(z_0, w_0, \zeta_0, \sigma(0))$ .

Now, define  $z_1 := \mathbb{E} \exp(\zeta_0)$ , and note that since  $z_0 \in \Omega_0^1$ ,  $z_1 \in \Omega_1^0$ . Thus, we can repeat the above process accordingly (either  $z_1 \in \mathcal{A}_1$  or  $z_1 \in \mathcal{B}_1$ ) to define the next quadruple  $(z_1, w_1, \zeta_1, \sigma(1))$ , and so on. In general, for every  $l \geq 0$ , we have

$$(8) \quad z_l = \mathbb{E} \exp(\zeta_{l-1}), \quad z_l \in \Omega_l^0, \quad f_l^{\circ \sigma(l)}(w_l) = z_l, \quad \Phi_l(w_l) := \zeta_l, \quad \text{and } 0 \leq \sigma(l) \leq k_l + \mathbf{k}'.$$

Note that by the definition of this sequence, for every  $l \geq 0$  we have

$$\text{either } \mathbf{k}' + 1/2 \leq \operatorname{Re} \zeta_l \leq \alpha_l^{-1} - \mathbf{k}, \text{ or } \zeta_l \in \cup_{j=1}^{\mathbf{k}'} B(j, \delta_3).$$

By Kőbe distortion theorem, there exists a constant  $\varepsilon > 0$  such that every map  $h \in \mathcal{IS}_{(0, \alpha^*]}$  is univalent on the ball  $B(0, \varepsilon)$ . Choose a constant  $D \geq \delta_3$  satisfying

$$(9) \quad \mathbb{E} \exp(Di) \leq \varepsilon.$$

Since  $\mathcal{R}h$  also belongs to  $\mathcal{IS}_{(0, \alpha^*]}$ , by the definition of renormalization,  $h^{k_h}$  is univalent on  $\{z \in S_h \mid \operatorname{Im} \Phi_h(z) \geq D\}$ . With this choice of  $D$ , we decompose the intersection into two sets

$$\begin{aligned} \Lambda &:= \{z \in \cap_{n=0}^{\infty} \Omega_0^n \cap J(P_\alpha) \mid \text{there are infinitely many } m \text{ with } \operatorname{Im} \zeta_m < D/\alpha_m\}, \\ \Gamma &:= \{z \in \cap_{n=0}^{\infty} \Omega_0^n \cap J(P_\alpha) \mid \exists N \text{ such that for all } m \geq N, \operatorname{Im} \zeta_m \geq D/\alpha_m\}. \end{aligned}$$

The area of these two sets are treated in the following two subsections.

**3.2. Area of  $\Gamma$ .** — The set  $\Gamma$  is contained in the union of the sets

$$\Gamma^m := \{z \in \Gamma \mid \text{for all } n \geq m, \operatorname{Im} \zeta_n \geq \frac{D}{\alpha_n}\}, \text{ for } m = 0, 1, 2, \dots$$

Every  $\Gamma^m$  projects (under piecewise holomorphic or anti-holomorphic maps) onto the set  $\Gamma_m$  on the dynamic plane of level  $m$ ; defined as

$$\Gamma_m := \{\operatorname{Exp}(\zeta_{m-1}) \mid z \in \Gamma^m\}.$$

Now fix an  $m \in \{0, 1, 2, \dots\}$  and for every  $j \geq m$  let

$$\Pi_j := \{\zeta_j + \sigma(j) \mid z \in \Gamma^m\}.$$

Projecting the above set onto the round cylinder  $\mathbb{C}/\mathbb{Z}$ , we define

$$B_j := \Pi_j/\mathbb{Z}, \text{ for } j = m, m+1, m+2, \dots$$

That is,  $B_j$  is the set of all points  $w \in \mathbb{C}/\mathbb{Z}$  for which there exists an integer  $i$  with  $w + i \in \Pi_j$ . Each  $B_j$  lies above the horizontal line passing through  $D\mathbf{i}/\alpha_j$ . We would like to show that these sets have zero area. To prove this, we first show that the sequence,

$$b_j := \sup_{\zeta, \zeta' \in B_j} \operatorname{Im}(\zeta - \zeta'), \text{ for } j = m, m+1, m+2, \dots,$$

satisfies the inequality in the following lemma.

**Lemma 3.3.** — *There exists a constant  $A$  such that for every  $j > m$  we have*

$$b_{j-1} \leq \alpha_j b_j + A.$$

To prove the above lemma, we need an estimates on the derivative of the change of coordinates from a level of renormalization to its previous level. To state the estimate, we slightly change the domain of definition of the perturbed Fatou coordinate. Indeed, given  $h \in \mathcal{IS}_{(0, \alpha^*]}$ , the map  $\Phi_h$ , considered on  $\{z \in \mathcal{P}_h \mid \operatorname{Re} \Phi_h(w) \geq k'\}$ , has a unique extension to the map

$$(10) \quad \Phi_h^\dagger : \mathcal{D}_h := \bigcup_{j=0}^{k_h + \lfloor 1/\alpha \rfloor - k - 1} h^{\circ j}(S_h) \setminus \{z \in \mathcal{P}_h \mid \operatorname{Re} \Phi_h(z) = k'\} \rightarrow \mathbb{C}.$$

It is defined using the dynamics of  $h$ , as follows. If  $z \in \mathcal{P}_h$  and  $\operatorname{Re} \Phi_h(z) > k'$ , we let  $\Phi_h^\dagger := \Phi_h(z)$ . Otherwise, for  $z \in \mathcal{D}_h$ , pick an integer  $j$  with  $1 \leq j \leq \lfloor 1/\alpha \rfloor + k_h - k - 1$  such that  $h^{-j}(z) \in S_h$  and  $\operatorname{Re} \Phi_h(h^{-j}(z)) > k'$ , then let  $\Phi_h^\dagger(z) := \Phi_h(h^{-j}(z)) + j$ . It follows from the equivariance property of  $\Phi_h$  that  $\Phi_h^\dagger$  given above is well-defined and continuous.

Let

$$(11) \quad \chi_h := \operatorname{Log} \circ (\Phi_h^\dagger)^{-1},$$

where  $\operatorname{Log}$  is an arbitrary inverse branch of  $\operatorname{Exp}$  defined on a slit neighborhood of 0.

For every  $r \in (0, 1/2]$  and  $\alpha \in (0, 1)$  define the set

$$\Theta(r, \alpha) := \{w \in \mathbb{C} \mid \operatorname{Im} w \geq -2/\alpha\} \setminus \bigcup_{n \in \mathbb{Z}} B(n/\alpha, r/\alpha).$$

**Lemma 3.4.** — *There exists a constant  $C$  such that for every  $r \in (0, 1/2]$ , every  $h \in \mathcal{IS}_{(0, \alpha^*]}$ , and every  $w \in \text{Dom } \chi_h \cap \Theta(r, \alpha)$ , we have*

$$|\chi'_h(w) - \alpha| \leq C \frac{\alpha}{r} e^{-2\pi\alpha \text{Im } w}.$$

This lemma is proved in Section 5.

*Proof of Lemma 3.3.* — Let  $\Phi_n^\dagger := \Phi_{f_n}^\dagger$  and  $\chi_n := \chi_{f_n}$ , for  $n = 0, 1, 2, \dots$ , denote the maps defined in Equations (10) and (11) when  $h = f_n$ . Since  $\text{Im } \zeta_j \geq D/\alpha_j \geq \delta_3$ , one infers from the definition of the quadruples that  $\Phi_j^\dagger(z_j) := \zeta_j + \sigma(j)$ . Therefore, from (8), for  $j = m, m+1, m+2, \dots$ , we have

$$\begin{aligned} \Pi_j &\subseteq \{w \in \mathbb{C} \mid \text{Im } w \geq D/\alpha_j, \mathbf{k}' + 1/2 \leq \text{Re } w \leq \mathbf{k}' + 1/2 + 1/\alpha_j - \mathbf{k} + k_n\}, \\ \chi_j(\Pi_j)/\mathbb{Z} &= B_{j-1}. \end{aligned}$$

For arbitrary  $\zeta, \zeta' \in \Pi_j$ , let  $\gamma$  be a straight line segment connecting  $\zeta'$  to  $\zeta$ . From Lemma 3.4, with  $r = \min\{1/2, D\}$ , we have

$$\begin{aligned} &|\chi_j(\zeta') - \chi_j(\zeta) - \alpha_j(\zeta' - \zeta)| \\ &= \left| \int_\gamma \chi'_j - \alpha_j dw \right| \\ &\leq \int_\gamma C \frac{\alpha_j}{r} e^{-2\pi\alpha_j \text{Im } w} dw \\ &\leq C \frac{\alpha_j}{r} e^{-2\pi\alpha_j(D/\alpha_j)} \left( \frac{1}{\alpha_j} - \mathbf{k} + k_n \right) + \int_{D/\alpha_j}^\infty C \frac{\alpha_j}{r} e^{-2\pi\alpha_j t} dt \\ &\leq \frac{C(1 + \mathbf{k}'')}{r} e^{-2\pi D} + \frac{e^{-2\pi D}}{2\pi r}. \end{aligned}$$

The result follows from taking supremum over all  $\chi_j(\zeta), \chi_j(\zeta')$ , and then all  $\zeta, \zeta'$ .  $\square$

**Lemma 3.5.** —  $\limsup_{j \geq m} b_j < 4A$ .

*Proof.* — For every  $j \geq 0$  recall the Brjuno sum

$$\mathcal{B}(\alpha_j) := \log \frac{1}{\alpha_j} + \alpha_j \log \frac{1}{\alpha_{j+1}} + \alpha_j \alpha_{j+1} \log \frac{1}{\alpha_{j+2}} + \dots$$

By a result of Yoccoz, [Yoc95], the Siegel disk of  $f_j$  contains the disk of radius  $e^{-\mathcal{B}(\alpha_j)+M}$  about the origin, for some universal constant  $M$ . This implies that for every  $j \geq m$ , we have  $b_j \leq \mathcal{B}(\alpha_{j+1})/2\pi$ .

Now fix  $j \geq m$ , and let  $l$  be an arbitrary integer bigger than  $j$ . Inductively using the inequality in Lemma 3.3, for  $j+1, j+2, \dots, l$ , we obtain

$$b_j \leq \alpha_{j+1} \alpha_{j+2} \cdots \alpha_l b_l + (1 + \alpha_{j+1} + \alpha_{j+1} \alpha_{j+2} + \cdots + \alpha_{j+1} \alpha_{j+2} \cdots \alpha_{l-1}) A.$$

Replacing  $b_l$  by  $\mathcal{B}(\alpha_{l+1})/2\pi$ , and that  $\alpha_i \alpha_{i+1} \leq 1/2$ , for all  $i$ , we come up with

$$b_j \leq \frac{1}{2\pi} \alpha_{j+1} \alpha_{j+2} \cdots \alpha_l \mathcal{B}(\alpha_{l+1}) + 4A.$$

As  $l$  tends to infinity, the first term in the above sum, which is the tail of the Brjuno series for  $\alpha_{j+1}$ , tends to zero.  $\square$

**Lemma 3.6.** — *There exists a constant  $\varepsilon > 0$  such that if  $\alpha_j < \varepsilon$  for some  $j \geq m$ , then  $\text{area } B_{j-1} < \frac{1}{2} \text{area } B_j$ .*

*Proof.* — Using Lemma 3.4 with  $h = f_j$ ,  $\chi_j := \chi_{f_j}$ , and  $r = \min\{1/2, D\}$ , we have,

$$\begin{aligned} \text{area } B_{j-1} &= \text{area } \chi_j(\Pi_j) \\ &\leq \int_{\Pi_j} |\chi'_j|^2 \frac{dz \wedge d\bar{z}}{2} \\ &\leq \left(\frac{1}{\alpha_j} + \mathbf{k}''\right) \cdot \max_{\ell \in \mathbb{Z}} \int_{\{w \in \Pi_j | \ell \leq \text{Re } w \leq \ell+1\}} |\chi'_j|^2 \frac{dz \wedge d\bar{z}}{2} \\ &\leq \left(\frac{1}{\alpha_j} + \mathbf{k}''\right) \left(C \frac{\alpha_n}{r} e^{-2\pi D}\right)^2 \text{area } B_j \\ &\leq \frac{1}{2} \text{area } B_j, \end{aligned}$$

with the last inequality for small enough values of  $\alpha_j$ .  $\square$

**Proposition 3.7.** — *The set  $\Gamma$  has zero area.*

*Proof.* — As  $\Gamma = \cup_{m \geq 0} \Gamma^m$ , it is enough to show that each  $\Gamma^m$  has zero area. Clearly,  $\Gamma^m$  has zero area if and only if  $\Gamma_m$  has zero area. To prove this, we show that  $B_m$  has zero area in the two separate cases.

*Case i:* Eventually  $\alpha_j \geq \varepsilon$ , where  $\varepsilon$  is obtained in Lemma 3.6.

For these values of  $\alpha$ ,  $\mathcal{B}(\alpha_j)$  is uniformly bounded, and hence, the set  $\Gamma$  is empty for large enough  $D$ . (In this case  $\alpha$  is of bounded type, and the post-critical set is known to be equal to the boundary of the Siegel disk and has zero area.)

*Case ii:* There are arbitrarily large  $j$  with  $\alpha_j < \varepsilon$ .

By Lemma 3.5,  $\text{area } B_j \leq 4A$ , for all  $j$ . On the other hand, there are infinitely many levels  $j$  with  $\text{area } B_{j-1} < \frac{1}{2} \text{area } B_j$ .  $\square$

### 3.3. Area of $\Lambda$ . —

**Proposition 3.8.** — *The set  $\cap_{n=0}^{\infty} \Omega_0^n \cap J(P_\alpha)$  is non-uniformly porous at every point in  $\Lambda$ . In particular,  $\Lambda$  has zero area.*

**Lemma 3.9.** — *There exists a constant  $E$  such that for every  $z \in \Lambda$  there are infinitely many positive integers  $m$  satisfying*

$$(12) \quad \text{Im } \zeta_m \leq \frac{1}{2\pi} \log \frac{1}{\alpha_{m+1}} + E.$$

This is also proved in Section 5.

The proof of the above proposition is exactly the one given at Subsection 4.3 in Part I. We only very briefly explain the idea.

For an arbitrary  $n$  satisfying the inequality in the above lemma, let

$$\mathcal{A}_n := B_{\delta_1}(B(\gamma_n(1), r^*) \cup \gamma_n[0, 1])$$



with  $\gamma_n$  and  $r^*$  introduced in Lemma 3.2. There exists a holomorphic or anti-holomorphic map

$$\mathcal{G}_n : \mathcal{A}_n \rightarrow \Omega_0^0,$$

defined as an appropriate composition of the maps  $\Phi_j^{-1}$ ,  $f_j$ , and the lifts  $\mathbb{L}og$ . We showed that the maps  $\mathcal{G}_n$  satisfy the following Lemma.

**Lemma 3.10.** — *There exist positive constants  $D_3$  and  $\delta_4 \in (0, 1)$  such that for every  $n$  satisfying (12) the map  $\mathcal{G}_n$  has the following properties*

- (1)  $\mathcal{G}_n(B(\gamma_n(1), r^*)) \cap \Omega_0^{n+1} = \emptyset$ ;
- (2) *there exists a positive constant  $r_n$  with*

$$B(\mathcal{G}_n(\gamma_n(1)), r_n) \subset \mathcal{G}_n(B(\gamma_n(1), r^*)) \quad \text{and} \quad |\mathcal{G}_n(\gamma_n(1)) - z_0| \leq D_3 \cdot r_n;$$

- (3)  $r_n \leq D_3(\delta_4)^n$ .

*Proof of Proposition 3.8.* — Let  $z_0$  be an arbitrary point in  $\Lambda$  so that we have a strictly increasing sequence  $n_i$  for which Inequality (12) holds. Lemma 3.2 introduces the balls  $B(\gamma_{n_i}(1), r^*)$  enjoying the properties in that lemma. By Lemma 3.10, the maps  $\mathcal{G}_{n_i}$  provide us with a sequence of balls  $B(\gamma_{n_i}(1), r_{n_i})$  satisfying

$$B(\gamma_{n_i}(1), r_{n_i}) \cap \Omega_{n_i+1}^0 = \emptyset, \quad |\mathcal{G}_{n_i+1}(\gamma_{n_i}(1)) - z_0| \leq D_3 \cdot r_{n_i}, \quad \text{and} \quad r_{n_i} \rightarrow 0.$$

The zero area statement follows from the Lebesgue density Theorem.  $\square$

**3.4. Non-uniform porosity of decorations.** — For Brjuno values of  $\alpha$ , let  $\Delta_\alpha$  denote the Siegel disk of  $P_\alpha$ , and  $\overline{\Delta}_\alpha$  denote its closure. Here we show that  $\mathcal{PC}(P_\alpha)$  is non uniformly porous at every point in  $\mathcal{PC}(P_\alpha) \setminus \Delta_\alpha$ .

**Lemma 3.11.** — *We have  $\mathcal{PC}(P_\alpha) \setminus \overline{\Delta}_\alpha \subset \Lambda$ .*

*Proof.* — Let  $z \in \mathcal{PC}(P_\alpha) \setminus \overline{\Delta}_\alpha$  and assume on the contrary that there is an integer  $N$  with  $\text{Im } \zeta_m \geq D/\alpha_m$ , for all  $m \geq N$ .

By the first argument in the proof of Lemma 3.5, we have

$$d(\zeta_j, \Phi_j^\dagger(\partial\Delta_j)) \leq \mathcal{B}(\alpha_{j+1})/2\pi,$$

for all  $j$ . The boundary of the Siegel disk of a map is sent to the boundary of the Siegel disk of its renormalization under the change of coordinates. Now it follows from Lemmas 3.3 and 3.5 that for all  $j \geq N$  we have  $d(\zeta_j, \Phi_j^\dagger(\partial\Delta_j)) \leq 4A$ . By a contraction argument for the changes of coordinates from a level of renormalization to the previous level with respect to appropriate hyperbolic metrics (see [Che10] section 4.3 for further details), one can see that all these distances must be zero. This implies that  $z \in \overline{\Delta}_\alpha$ .  $\square$

Theorem D follows from the above lemma and Proposition 3.8.

#### 4. Accumulation on the critical point

Here we introduce a sequence of subsets of  $\mathbb{C}$  containing  $\text{cp}$ ,  $Q_0^n$ , that are visited by the orbit of almost every  $z \in J(P_\alpha)$ .

Recall the sets  $\mathcal{C}_n^{-k_n}$  and  $(\mathcal{C}_n^\sharp)^{-k_n}$  obtained in the definition of  $\mathcal{R}(f_n)$ . For every  $n \geq 0$ , define the sets

$$W_n := \mathcal{C}_n^{-k_n} \subset S_n^0, \text{ and}$$

$$W^n := \Psi_n(W_n) \subset S_0^n$$

Note that for every  $n \geq 0$  there is a pre-critical point of  $f_n$  in  $W_n$  and hence, by Lemma 2.2-(2), there is a pre-critical point of  $f_0$  in  $W^n$ . For every  $n \geq 0$ , define

$$Q^n := f_0^{\circ \tau(n)}(W^n),$$

where  $\tau(n)$  is the smallest positive integer with  $\text{cp} \in \text{int}(f_0^{\circ \tau(n)}(W^n))$ .

**Lemma 4.1.** — *For all  $n \geq 0$  and all  $z \in (\mathcal{C}_n^\sharp)^{-k_n}$ , we have  $\text{Im } \phi_n(z) \geq -2$ .*

*Proof.* — By the definition of the renormalization  $\mathcal{R}f_n$ , it is enough to show that

$$\forall w \in B(0, \frac{4}{27}e^{-4\pi}), \text{ we have } f_{n+1}^{-1}(w) \in B(0, \frac{4}{27}e^{4\pi}),$$

where  $f_{n+1}^{-1}(w)$  is the unique pre-image of  $w$ . The map  $f_{n+1}$  has a unique simple critical point mapped to  $-4/27$ . Hence, there is an inverse branch of  $f_{n+1}$  defined on the ball  $B(0, -4/27)$ . By K oebe distortion Theorem, it easily follows that  $f_{n+1}^{-1}(B(0, \frac{4}{27}e^{-4\pi}))$  is contained in  $B(0, \frac{4}{27}e^{4\pi})$ .  $\square$

**Lemma 4.2.** — *Assume that  $z \in \Omega^n \setminus \Omega^{n+1}$  and  $f_0(z) \in \Omega^{n+1}$ , for some  $n \geq 0$ . Then, either  $z \in Q^n$  or  $f_0(z) \in W^{n+1}$  (hence,  $f_0^{\circ(\tau(n+1)+1)}(z) \in Q^{n+1}$ ).*

*Proof.* — There is a

$$j \in \{0, 1, \dots, q_{n+1}(k_{n+1} + \lfloor 1/\alpha_{n+1} \rfloor - k - 1) + q_n\},$$

such that  $f_0(z) \in f_0^{\circ j}(S_0^{n+1})$ , by the definition of  $\Omega^{n+1}$ . If  $j \neq 0$ , there is a  $w \in f_0^{j-1}(S_0^{n+1})$  with  $f_0(w) = f_0(z)$ . If  $z \notin Q^n$ , as  $f_0 : \Omega^n \setminus Q^n \rightarrow \mathbb{C}$  is univalent, we must have  $z = w \in \Omega^{n+1}$  which contradicts the assumption in the lemma. Therefore, either  $j = 0$  or  $z \in Q^n$ .

If  $j = 0$ , then

$$\begin{aligned} f_0(z) &\in S_0^{n+1} = \Psi_{n+1}(S_{n+1}^0) \\ &= W^{n+1} \cup \Psi_{n+1}((\mathcal{C}_{n+1}^\sharp)^{-k_{n+1}}) \end{aligned}$$

However, if  $f_0(z) \in \Psi_{n+1}((\mathcal{C}_{n+1}^\sharp)^{-k_{n+1}})$ , by Lemma 4.1,  $z \in \Omega^{n+1}$ . This finishes the proof of the lemma.  $\square$

Let us define the *eccentricity* of a domain  $Q$  at a point  $q \in \text{int } Q$  as

$$\text{Ecc}(Q, q) := \frac{\inf\{r \mid Q \subset B(q, r)\}}{\sup\{r \mid B(q, r) \subset Q\}}.$$

**Lemma 4.3.** — *We have*

- $\sup_n \text{Ecc}(Q^n, \text{cp}) < \infty$ ,
- $\lim_{n \rightarrow \infty} \text{diam } Q^n = 0$ .

*Proof.* — The proof of the first statement is broken into four steps.

*Step 1.* For every  $n \geq 0$ , there exists a simply connected domain  $\widehat{W}_n$  containing  $W_n$  with the following properties.

- $\widehat{W}_n \subset \mathcal{P}_n$ ,  $f_n^{\circ k_n}(\widehat{W}_n) \subset \mathcal{P}_n$ ,
- $\inf_n \text{mod}(\widehat{W}_n \setminus W_n) > 0$ ,
- $f_n^{\circ k_n} : \widehat{W}_n \rightarrow f_n^{k_n}(\widehat{W}_n)$  is a proper branched covering of degree two.

Recall that  $f_n^{\circ k_n} : W_n \rightarrow \mathcal{C}_n$  is a proper branched covering of degree two,  $k_n$  is uniformly bounded (Lemma 1.3),  $B_{1/2}(W_n) \subset \mathcal{P}_n$ , and  $B_{1/2}(\mathcal{C}_n) \subset \mathcal{P}_n$ . Now the above step follows from the compactness of the class  $\mathcal{IS}$ , and the continuous dependence of  $W_n$  on  $f_n$ .

*Step 2.* If  $f_n^{-k_n}(\text{cv}_{f_n})$  is the unique preimage of  $\text{cv}_{f_n}$  contained in  $W_n$  then,

$$\sup_n \text{Ecc}(W_n, f_n^{-k_n}(\text{cv}_{f_n})) < \infty.$$

By Kőbe distortion Theorem,  $\sup_n \text{Ecc}(\mathcal{C}_n, \text{cv}_{f_n}) < \infty$ . Indeed,  $\Phi_n^{-1}$  is defined and univalent on the  $1/2$ -neighborhood of  $\Phi_n(\mathcal{C}_n)$ . The map  $f_n^{\circ k_n} : W_n \rightarrow \mathcal{C}_n$  belongs to a compact class by Step 1. This implies the above statement on eccentricity of  $W_n$ .

*Step 3.*  $f_0^{\circ \tau(n)}$  is univalent on  $\Psi_n(\widehat{W}_n)$ , and  $f_0^{\circ \tau(n)}(\Psi_n(f_n^{-k_n}(\text{cv}_{f_n}))) = \text{cp}$ .

By Step 1 and Lemma 2.2, the map  $f_0^{\circ(k_n q_n + q_{n-1})} : \Psi_n(\widehat{W}_n) \rightarrow \mathcal{P}_0$  is a proper branched covering of degree two mapping  $\Psi_n(f_n^{-k_n}(\text{cv}_{f_n}))$  to  $\text{cv}$ . We must have

$$(13) \quad \tau(n) < k_n q_n + q_{n-1}.$$

On the other hand,  $f_0^{\circ \tau(n)}(W^n)$  contains a critical point, and therefore,  $f_0^{\circ \tau(n)}$  must be univalent on  $\Psi_n(\widehat{W}_n)$ .

*Step 4.* We have  $\sup_n \text{Ecc}(Q^n, \text{cp}) < \infty$ .

By definition  $Q^n := f_0^{\circ \tau(n)} \circ \Psi_n(W_0^n)$ , and by Step 3,  $f_0^{\circ \tau(n)} \circ \Psi_n$  has univalent extension onto  $\widehat{W}_n$  with  $\inf_n \text{mod}(\widehat{W}_n \setminus W_n) > 0$ . Again by Kőbe distortion Theorem, there exists a constant  $C(\varepsilon)$  such that

$$\text{Ecc}(Q^n, \text{cp}) \leq C(\varepsilon) \text{Ecc}(W_n, f_n^{-k_n}(\text{cv}_{f_n})).$$

Now Step 4 follows from Step 2.

To prove the second part, we use Montel's normal family Theorem. Since  $Q^n \subset f_0^{\circ \tau(n)}(S_0^n)$ , by definition of  $\Omega^n$ ,  $f_0$  can be iterated at least

$$q_n(k_n + \lfloor 1/\alpha_n \rfloor - k - 1) + q_{n-1} - \tau(n)$$

times on  $Q^n$  with values in  $\Omega^n$ . By (13), the above quantity goes to infinity as  $n$  goes to infinity.

Now if  $\limsup_{n \rightarrow \infty} \text{diam}(Q^n) \neq 0$ , by the first part, a ball of positive size about  $\text{cp}$  is contained in infinitely many  $Q^n$ 's. By Montel's normal family Theorem, this implies that the critical point belongs to the Fatou set of  $f_0$ , which is not possible.  $\square$

Indeed, one can show that  $\text{diam } Q^n$  converges exponentially fast to zero by showing that the changes of coordinates between different levels of renormalization are uniformly contracting. See Subsection 4.3 in [Che10].

*Proof of Theorem C.* — By the argument after Proposition 2.3, the sets

$$E_n := \{z \in J(P_\alpha) \mid \mathcal{O}(z) \text{ eventually stays in } \Omega^n\},$$

for  $n \geq 0$ , have full Lebesgue measure in  $J(P_\alpha)$ . Therefore, their intersection has full Lebesgue measure in  $J(P_\alpha)$ .

Let  $z \in \bigcap_{n=0}^{\infty} E_n \cap J(P_\alpha)$ . As  $\bigcap_{j=0}^{\infty} \Omega^n \cap J(P_\alpha)$  has zero area by Proposition 3.1 in the introduction and Theorem C in [Che10], there are increasing sequences of positive integers  $\{n_i\}_{i=1}^{\infty}$  and  $\{t_i\}_{i=1}^{\infty}$  with the following property

$$P_\alpha^{\circ t_i}(z) \in \Omega^{n_i} \setminus \Omega^{n_i+1}, \text{ and } P_\alpha^{\circ t_i+1}(z) \in \Omega^{n_i+1}.$$

Now by Lemma 4.2, there are positive integers  $s_i \geq t_i$  with  $P_\alpha^{\circ s_i}(z) \in Q_0^{n_i}$ . Combining with Lemma 4.3, we conclude that the orbit of  $z$  gets arbitrarily close to the critical point of  $P_\alpha$ .  $\square$

## 5. Perturbed Fatou coordinate

**5.1. The Pre-Fatou coordinate.** — Let

$$h(z) = e^{2\pi\alpha i} \cdot P \circ \varphi^{-1}(z) : \varphi(U) \rightarrow \mathbb{C}$$

be a map in the class  $\mathcal{IS}_\alpha$  with  $\alpha \neq 0$ , and let  $\sigma_h$  denote the non-zero fixed point of  $h$ . For  $\alpha$  small enough,  $h(z)$  has only two fixed points 0 and  $\sigma_h$  in a fixed neighborhood of 0 (This property is guaranteed by assuming  $\alpha \in \text{Irrat}_{\geq N}$ , [IS06]). By K oebe distortion Theorem one can see that  $\text{cp}_h \in B(0, 2) \setminus B(0, 0.22)$ .

Consider the universal covering

$$\tau_h(w) := \frac{\sigma_h}{1 - e^{-2\pi i \alpha w}} : \mathbb{C} \rightarrow \hat{\mathbb{C}} \setminus \{0, \sigma_h\}$$

that satisfies  $\tau_h(w + \alpha^{-1}) = \tau_h(w)$ , for all  $w \in \mathbb{C}$ ,  $\tau_h(w) \rightarrow 0$  as  $\text{Im}(\alpha w) \rightarrow \infty$ , and  $\tau_h(w) \rightarrow \sigma_h$  as  $\text{Im}(\alpha w) \rightarrow -\infty$ .

The map  $h$  lifts under the covering  $\tau_h$  to the map

$$F_h(w) := w + \frac{1}{2\pi\alpha i} \log \left( 1 - \frac{\sigma_h u_h(z)}{1 + z u_h(z)} \right), \quad \text{with } z = \tau_\alpha(w), u_h(z) := \frac{h(z)}{z(z - \sigma_h)}$$

defined on the set of points  $w$  with  $\tau_h(w) \in \text{Dom } h$ . The branch of  $\log$  in the above formula is determined by  $-\pi < \text{Im} \log(\cdot) < \pi$ . We have

$$(14) \quad h \circ \tau_h = \tau_h \circ F_h, \quad \text{and} \quad F_h(w) + \alpha^{-1} = F_h(w + \alpha^{-1}).$$

The plan is to estimate the conformal mapping  $L_h$  that conjugates  $F_h$  to the translation by one. For every real number  $R > 0$ , let  $\Theta(R)$  denote the set

$$\Theta(R) := \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} B(n/\alpha, R).$$

By explicit estimates on  $\tau_h$  and Taylor's Estimate on  $h$  we proved the following properties on  $F_h$  in Part I.

**Lemma 5.1.** — *There exist positive constants  $\varepsilon_0$ ,  $C_2$ ,  $C_3$ , and  $C_4$  such that for every map  $h \in \mathcal{IS}_\alpha$  with  $\alpha \leq \varepsilon_0$  we have*

- i. *The induced map  $F_h$  is defined and is univalent on  $\Theta(C_2)$ . Moreover, on  $\Theta(C_2)$*

$$|F_h(w) - (w + 1)| < 1/4, \quad \text{and} \quad |F'_h(w) - 1| < 1/4.$$

- ii. *For every  $r \in (0, 1/2)$  and every  $w \in \Theta(r/\alpha) \cap \Theta(C_2)$ , we have*

$$|F_h(w) - (w + 1)| < C_3 \frac{\alpha}{r} e^{-2\pi\alpha \operatorname{Im} w}.$$

- iii. *There exist a critical point  $\operatorname{cp}_{F_h} \in B(0, C_2)$  of  $F_h$ , and the smallest non-negative integer  $i(h)$  with*

$$F_h^{\circ i(h)}(\operatorname{cp}_{F_h}) \in \Theta(C_2), \quad \text{and} \quad \sup_h i(h) < \infty.$$

- iv. *For every positive integer  $j \leq i(h) + \frac{2}{3\alpha}$ , we have*

$$|F_h^{\circ j}(\operatorname{cp}_{F_h}) - j| \leq C_4(1 + \log j).$$

**5.2. The linearizing coordinate.** — Let  $h \in \mathcal{IS}_{(0, \alpha_*]}$  with the perturbed Fatou coordinate  $\Phi_h : \mathcal{P}_h \rightarrow \mathbb{C}$ . The inverse image  $\tau_h^{-1}(\mathcal{P}_h)$  has countably many components going from  $-\mathbf{i}\infty$  to  $+\mathbf{i}\infty$ . Define

$$(15) \quad L_h := \Phi_h \circ \tau_h : \operatorname{Dom} L_h \rightarrow \mathbb{C},$$

where  $\operatorname{Dom} L_h$  is, by definition, the component of  $\tau_h^{-1}(\mathcal{P}_h)$  separating 0 and  $1/\alpha$  in  $\mathbb{C}$ . It follows from Theorem 1.1 that the map  $L_h : \operatorname{Dom} L_h \rightarrow \mathbb{C}$  is univalent,  $L_h(\operatorname{cv}_{F_h}) = 1$ ,  $L_h(\operatorname{Dom} L_h)$  contains the set

$$\{w \in \mathbb{C} : 0 < \operatorname{Re}(w) < \lfloor 1/\alpha \rfloor - k\},$$

$L_h(w) \rightarrow +\mathbf{i}\infty$  as  $\operatorname{Im}(w) \rightarrow +\infty$ , and  $L_h(w) \rightarrow -\mathbf{i}\infty$  as  $\operatorname{Im}(w) \rightarrow -\infty$ . Moreover,  $L_h$  satisfies the Abel functional equation

$$(16) \quad L_h(F_h(w)) = L_h(w) + 1,$$

whenever both sides are defined.

Using (16), if  $\alpha < \varepsilon_0$ ,  $L_h$  extends to a univalent map on the set

$$\begin{aligned} \Sigma_{C_2} := & \{w \in \mathbb{C} : C_2 \leq \operatorname{Re}(w) \leq \alpha^{-1} - C_2\} \\ & \cup \{w \in \mathbb{C} : \operatorname{Re}(w) \geq \alpha^{-1} - C_2, \text{ and } |\operatorname{Im} w| \geq \operatorname{Re}(w) - \alpha^{-1} + 2C_2\} \\ & \cup \{w \in \mathbb{C} : \operatorname{Re}(w) \leq C_2, \text{ and } |\operatorname{Im} w| \geq -\operatorname{Re}(w) + 2C_2\}. \end{aligned}$$

More precisely, by Lemma 5.1-i, for every  $w \in \Sigma_{C_2}$  there is an integer  $j$  with  $F_h^j(w) \in \operatorname{Dom} L_h$ . One defines  $L_h(w) := L_h(F_h^j(w)) - j$ . We denote the extended map by the

same notation  $L_h$ . It follows from K oebe distortion Theorem and Equation (16) that the derivative of  $L_h$  is uniformly bounded from below and above (see Part I, Lemma 5.3, for details).

**Lemma 5.2.** — *There exists a positive constant  $C_5$  such that for every  $h \in \mathcal{IS}_\alpha$ , and every  $\zeta \in L_h(\text{Dom } L_h) \setminus B(0, 1/2)$  we have  $1/C_5 \leq |(L_h^{-1})'(\zeta)| \leq C_5$ . Moreover,  $L_h'(z) \rightarrow 1$ , as  $\text{Im } z \rightarrow \infty$ .*

**5.3. The model.** — Given  $A \in \mathbb{C}$ , let  $\mathcal{K}_h$  denoted the domain (with asymptotic width one) bounded by the curves  $A + ti$ ,  $F_h(A + ti)$ ,  $(1-s)A + sF_h(A)$ , for  $t \in [0, \infty)$  and  $s \in [0, 1]$ . Moreover, assume that  $\mathcal{K}_h$  is contained in  $\Theta(C_2 + 1) \cap \Theta(r/\alpha)$ .

For  $(s, t) \in [0, 1] \times [0, \infty)$  define the map

$$H(s, t) := A + \int_0^s X(\ell, t) d\ell + \mathbf{i}(t + \int_0^s Y(\ell, t) d\ell),$$

where  $X$  and  $Y$  are given as

$$X(s, t) := a_0(t) + a_1(t) \sin(\pi s) + a_2(t) \cos(\pi s) + a_3(t) \sin(2\pi s) + a_4(t) \cos(2\pi s),$$

$$Y(s, t) := b_0(t) + b_1(t) \sin(\pi s) + b_2(t) \cos(\pi s) + b_3(t) \sin(2\pi s) + b_4(t) \cos(2\pi s),$$

with

$$a_0(t) = \text{Re}(F_h(A + \mathbf{i}t) - A) + \text{Re } F_h''(A + \mathbf{i}t)/\pi,$$

$$b_0(t) = \text{Im}(F_h(A + \mathbf{i}t) - A) - t + \text{Im}(F_h''(A + \mathbf{i}t))/\pi,$$

$$a_1(t) = -\text{Re } F_h''(A + \mathbf{i}t)/2$$

$$b_1(t) = -\text{Im } F_h''(A + \mathbf{i}t)/2,$$

$$a_2(t) = (1 - \text{Re } F_h'(A + \mathbf{i}t))/2$$

$$b_2(t) = -\text{Im } F_h'(A + \mathbf{i}t)/2,$$

$$a_3(t) = \text{Re } F_h''(A + \mathbf{i}t)/4,$$

$$b_3(t) = \text{Im } F_h''(A + \mathbf{i}t)/4,$$

$$a_4(t) = 1 - a_0(t) - a_2(t),$$

$$b_4(t) = -a_0(t) - a_2(t)$$

(The above coefficients are obtained from solving a separable partial differential equation so that the next lemma holds.)

Instead of algebraically manipulating constants and frequently introducing new ones, we use the following convention from now on. Given real valued functions  $f$  and  $g$  defined on a set  $W$ , we write

$$f(x) \preceq g(x) \text{ on } W$$

to mean that there exists a constant  $M$  with  $f(x) \leq Mg(x)$  for all  $x \in W$ . For example, by Lemma 5.1 and Cauchy Integral Estimate, on the set  $\Theta(r/\alpha) \cap \Theta(C_2 + 1)$  we have

$$(17) \quad \max \{|F_h(w) - w - 1|, |F_h'(w) - 1|, |F_h''(w)|, |F_h^{(3)}(w)|, |F_h^{(4)}(w)|\} \preceq \frac{\alpha}{r} e^{-2\pi\alpha \text{Im } w}.$$

Using the relation  $H(s + 1, t) = F_h(H(s, t))$  we extend  $H$  onto a neighborhood of the form  $(-\delta, 1 + \delta) \times (0, \infty)$ , for some  $\delta > 0$ .

**Lemma 5.3.** — *The map  $H$  satisfies the following properties.*

i. For every  $t \in (0, \infty)$  and  $s \in (-\delta, \delta)$  we have

$$F(H(s, t)) = H(s + 1, t).$$

ii.  $H$  is  $C^2$  on  $(-\delta, 1 + \delta) \times (0, \infty)$ .

iii. On  $(-\delta, 1 + \delta) \times (0, \infty)$ , we have

$$\max \{ |\partial_s H(s, t) - 1|, |\partial_t H(s, t) - \mathbf{i}| \} \preceq \frac{\alpha}{r} e^{-2\pi\alpha(t + \operatorname{Im} A)},$$

and

$$\max \{ |\partial_{ss} H(s, t)|, |\partial_{tt} H(s, t)|, |\partial_{st} H(s, t)| \} \preceq \frac{\alpha}{r} e^{-2\pi\alpha(t + \operatorname{Im} A)}.$$

*Proof.* — Clearly  $H(0, t) = A + t\mathbf{i}$ . On the other hand

$$\begin{aligned} H(1, t) &= A + a_0(t) + 2a_1(t)/\pi + (t + b_0(t) + 2b_1(t)/\pi)\mathbf{i} \\ &= F(A + t\mathbf{i}) \end{aligned}$$

For other values of  $s$ , the relation follows from the definition.

The map  $H$  is real analytic in the domain  $(-\delta, 1 + \delta) \times (0, \infty) \setminus \{0, 1\} \times (0, \infty)$ . A straightforward calculation shows that it is  $C^2$  on the lines  $\{0, 1\} \times (0, \infty)$ .

From Inequality (17), on  $[0, \infty)$  we have

$$\begin{aligned} |a_0(t) - 1| &\preceq \frac{\alpha}{r} e^{-2\pi\alpha(t + \operatorname{Im} A)}, \\ |a_j(t)| &\preceq \frac{\alpha}{r} e^{-2\pi\alpha(t + \operatorname{Im} A)}, \text{ for } j = 1, 2, 3, 4, \\ |b_j(t)| &\preceq \frac{\alpha}{r} e^{-2\pi\alpha(t + \operatorname{Im} A)}, \text{ for } j = 0, 1, 2, 3, 4. \end{aligned}$$

Similar estimates hold for the first and second derivatives of these functions. This implies that on  $[0, 1] \times (0, \infty)$

$$\max \{ |X - 1|, |\partial_s X|, |\partial_t X|, |\partial_{tt} X|, |Y|, |\partial_t Y|, |\partial_{tt} Y| \}(s, t) \preceq \frac{\alpha}{r} e^{-2\pi\alpha(t + \operatorname{Im} A)}.$$

One infers the inequalities in Part iii from these estimates.  $\square$

To simplify the calculations, from now on we use the complex notation  $z = s + t\mathbf{i}$ ,  $dz = ds + \mathbf{i}dt$ ,  $d\bar{z} = ds - \mathbf{i}dt$ ,  $\partial_z = (\partial_s - \mathbf{i}\partial_t)/2$ ,  $\partial_{\bar{z}} = (\partial_s + \mathbf{i}\partial_t)/2$ .

To compare  $L_h^{-1}$  to  $H$  we work on the map

$$G := L_h \circ H : (-\delta, 1 + \delta) + \mathbf{i}(0, \infty) \rightarrow \mathbb{C}.$$

In particular, we would like to prove that

$$(18) \quad |\partial_z G(z) - 1| \preceq \frac{1}{r} e^{-2\pi\alpha \operatorname{Im} z}.$$

**Lemma 5.4.** — The map  $G$  has a  $C^2$  extension onto  $\mathbb{C}$  which satisfies

$$(19) \quad G(z + 1) = G(z) + 1, \forall z \in \mathbb{C},$$

and is the translation by  $G(0, 0)$  on  $\{z \mid \operatorname{Im} z \leq -1\}$ .

i. On  $\{z \mid \operatorname{Im} z \geq 0\}$ ,

$$\max\{|\partial_{\bar{z}}G(z)|, |\partial_{z\bar{z}}G(z)|\} \preceq \frac{\alpha}{r} e^{-2\pi\alpha \operatorname{Im}(z+A)},$$

and on  $\{z \mid \operatorname{Im} z \in [-1, 0]\}$ ,

$$\max\{|\partial_{\bar{z}}G(z)|, |\partial_{z\bar{z}}G(z)|\} \preceq 1.$$

- ii. There exists a constant  $z_0$  such that as  $\operatorname{Im} z \rightarrow \infty$ ,  $G(z) - z \rightarrow z_0$ .  
 iii. For every constant  $\delta \in \mathbb{R}$  there is a constant  $C_7$  independent of  $r$  such that if  $\mathcal{K}_h \subseteq \Theta(r/\alpha)$  and  $\liminf_{w \in \mathcal{K}_h} \operatorname{Im} w \geq \delta/\alpha$  then for every  $z_1, z_2 \in \operatorname{Dom} G$ ,

$$|G(z_2) - G(z_1) - (z_2 - z_1)| \leq C_7/r.$$

*Proof.* — From Lemma 5.3-i and functional Equation (16), the Relation (19) holds for every  $z$  with  $\operatorname{Re} z \in (-\delta, \delta)$  and  $\operatorname{Im} z > 0$ . Moreover, it is  $C^2$  as  $H$  is  $C^2$  smooth and  $L_h$  is analytic. We can use (19) to extend  $G$  onto the upper half plane.

Below the line  $\operatorname{Im} z = -1$  we define  $G(z) := z + G(0, 0)$ . Then, there exists a  $C^2$  smooth interpolation of  $G$  on the strip  $\operatorname{Im} z \in [-1, 0]$  that satisfies (19), and has uniformly bounded first and second order partial derivatives. (For example, one may obtain this interpolation by first  $C^1$  gluing the vector field  $\partial G/\partial s$  over the upper half plane to the constant vector field 1 below the line  $\operatorname{Im} z = -1$ , and then integrate this vector field.)

*Part i:* By complex chain rule, with  $w = H(z)$ , we have

$$\begin{aligned} |\partial_{\bar{z}}G(z)| &= |\partial_w L_h(H(z)) \cdot \partial_{\bar{z}}H(z)| \\ &\preceq \frac{\alpha}{r} e^{-2\pi\alpha \operatorname{Im}(z+A)}, \end{aligned}$$

by Lemmas 5.2 and 5.3-iii.

On the other hand, Lemma 5.2 and the Cauchy Integral Formula implies that  $|\partial_{ww}L_h|$  is uniformly bounded well inside the domain of  $L_h$ . Therefore, from the estimates in Lemma 5.3, and the complex chain rule

$$\begin{aligned} |\partial_{z\bar{z}}G(z)| &= |\partial_{ww}L_h(H(z)) \cdot \partial_z H(z) \cdot \partial_{\bar{z}}H(z) + \partial_w L_h(H(z)) \cdot \partial_{z\bar{z}}H(z)| \\ &\preceq \frac{\alpha}{r} e^{-2\pi\alpha \operatorname{Im}(z+A)}. \end{aligned}$$

*Part ii:* Since  $G$  is periodic of period one, it is enough to prove the asymptotic behavior at  $z$  with  $\operatorname{Re} z \in [0, 1]$ . In this region, it is enough to prove that for every  $\varepsilon > 0$  there is  $R \in \mathbb{R}$  such that if  $\operatorname{Im} z_1$  and  $\operatorname{Im} z_2$  are larger than  $R$  then  $|(G(z_2) - z_2) - (G(z_1) - z_1)|$  is less than  $\varepsilon$ .

Given real constants  $t_2 > t_1$ , define  $\mathcal{D} := \{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z \leq 1, t_1 \leq \operatorname{Im} z \leq t_2\}$ . By Green's Integral Formula, we have

$$(20) \quad \int_{\partial\mathcal{D}} G(z) dz = \iint_{\mathcal{D}} \partial_{\bar{z}}G(z) d\bar{z}dz.$$



The right hand integral is bounded by

$$\begin{aligned}
 \left| \iint_{\mathcal{D}} \partial_{\bar{z}} G(z) dz d\bar{z} \right| &\leq 2 \int_{t_1}^{t_2} \int_0^1 |\partial_{\bar{z}} G(s, t)| ds dt \\
 &\preceq \int_{t_1}^{t_2} \frac{\alpha}{r} e^{-2\pi\alpha(t+\text{Im } A)} ds dt \quad (\text{Lemma 5.4-ii}) \\
 &\leq \frac{e^{-2\pi\alpha \text{Im } A}}{2\pi r} e^{-2\pi\alpha t_1}.
 \end{aligned}$$

This bound tends to zero, as  $t_1$  goes to infinity.

By periodicity of  $G$ , the left hand side of (20) is equal to

$$-\mathbf{i}(t_2 - t_1) + \int_{\gamma_1} G(z) ds + \int_{\gamma_2} G(z) ds,$$

where  $\gamma_1$  and  $\gamma_2$  are the top and bottom boundaries of  $\mathcal{D}$  with appropriate orientations. By Lemmas 5.2 and 5.3-iii, as  $\text{Im } z \rightarrow \infty$ ,  $\partial_{\bar{z}} G(z) \rightarrow 1$ . Hence

$$\int_{\gamma_1} G(z) ds \rightarrow - \int_0^1 G(t_1 \mathbf{i}) + 1 \cdot s ds = -G(t_1 \mathbf{i}) - 1/2$$

and

$$\int_{\gamma_2} G(z) ds \rightarrow \int_0^1 G(t_2 \mathbf{i}) + 1 \cdot s ds = G(t_2 \mathbf{i}) + 1/2$$

Putting these together, we conclude that as  $\text{Im } z_1, \text{Im } z_2 \rightarrow \infty$ ,

$$\begin{aligned}
 |(G(z_2) - z_2) - (G(z_1) - z_1)| \\
 \rightarrow |(G(\mathbf{i} \text{Im } z_2) - \mathbf{i} \text{Im } z_2) - (G(\mathbf{i} \text{Im } z_1) - \mathbf{i} \text{Im } z_1)| \rightarrow 0.
 \end{aligned}$$

This finishes the proof of this part.

*Part iii:* This follows from the above relations and inequalities once we have a lower bound on  $\text{Im } z_1$  and  $\text{Im } z_2$ . (See Part I, proof of Lemma 5.4 for further details.)  $\square$

The map  $G$  projects via  $e^{2\pi \mathbf{i} z}$  to a well-defined map on  $\mathbb{C}$ . More precisely, let

$$\phi(z) := e^{2\pi \mathbf{i} z}, \psi(\zeta) := \phi^{-1}(\zeta) = \frac{1}{2\pi \mathbf{i}} \log \zeta, \text{ with } \text{Im } \log(\cdot) \in [0, 2\pi),$$

and define

$$K(\zeta) := \phi \circ G \circ \psi(\zeta).$$

The map  $K$  has continuous extension to 0;  $K(0) = 0$ .

**Lemma 5.5.** — *The map  $K : \mathbb{C} \rightarrow \mathbb{C}$  is complex differentiable at 0, and*

$$|\partial_{\zeta} K(\zeta) - \partial_{\zeta} K(0)| \preceq \frac{1}{r} |\zeta|^{\alpha}.$$

*Proof.* — By the definition of derivative and Lemma 5.4-iii,

$$\begin{aligned}
 \partial_\zeta K(0) &= \lim_{\zeta \rightarrow 0} \frac{e^{2\pi i G(\frac{1}{2\pi i} \log \zeta)}}{\zeta} \\
 (21) \quad &= \lim_{\zeta \rightarrow 0} e^{2\pi i (G(\frac{1}{2\pi i} \log \zeta) - \frac{1}{2\pi i} \log \zeta)} \\
 &= e^{2\pi i z_0}.
 \end{aligned}$$

To prove the inequality, first we estimate the Laplacian of  $K$  on  $\mathbb{C} \setminus \{0\}$ . Using the complex chain rule, with  $\zeta = \phi(z)$  and  $\psi(\zeta) = z$ , at  $\zeta \neq 0$

$$\begin{aligned}
 \partial_\zeta K &= \partial_\zeta (\phi \circ G \circ \psi) = \partial_z \phi \circ (G \circ \psi) \cdot \partial_\zeta (G \circ \psi) + \partial_{\bar{z}} \phi \circ (G \circ \psi) \cdot \partial_{\bar{\zeta}} (\overline{G \circ \psi}) \\
 (22) \quad &= \partial_z \phi \circ G \circ \psi \cdot \partial_\zeta (G \circ \psi) \\
 &= \partial_z \phi \circ G \circ \psi \cdot \partial_z G \circ \psi \cdot \partial_\zeta \psi,
 \end{aligned}$$

therefore,

$$\begin{aligned}
 \partial_{\bar{\zeta}} \partial_\zeta K &= \partial_{\bar{\zeta}} (\partial_z \phi \circ (G \circ \psi)) \cdot \partial_z G \circ \psi \cdot \partial_\zeta \psi \\
 &\quad + \partial_z \phi \circ (G \circ \psi) \cdot \partial_{\bar{\zeta}} (\partial_z G \circ \psi) \cdot \partial_\zeta \psi \\
 &= (\partial_{zz} \phi \circ G \circ \psi \cdot \partial_{\bar{z}} G \circ \psi \cdot \partial_{\bar{\zeta}} \bar{\psi} \cdot \partial_z G \circ \psi) \cdot \partial_z \psi \\
 &\quad + \partial_z \phi \circ G \circ \psi \cdot ((\partial_{zz} G \circ \psi) \cdot \partial_{\bar{\zeta}} \bar{\psi} + \partial_{z\bar{z}} G \circ \psi \cdot \partial_{\bar{\zeta}} \bar{\psi}) \cdot \partial_z \psi \\
 &= \partial_{zz} \phi \circ G \circ \psi \cdot \partial_{\bar{z}} G \circ \psi \cdot \overline{\partial_{\bar{\zeta}} \bar{\psi}} \cdot \partial_z G \circ \psi \cdot \partial_z \psi \\
 &\quad + \partial_z \phi \circ G \circ \psi \cdot \partial_{z\bar{z}} G \circ \psi \cdot \overline{\partial_{\bar{\zeta}} \bar{\psi}} \cdot \partial_z \psi \\
 &= 2\pi i K |\partial_\zeta \psi|^2 (\partial_{z\bar{z}} G \circ \psi + 2\pi i \partial_{\bar{z}} G \circ \psi \cdot \partial_z G \circ \psi).
 \end{aligned}$$

(To get the above expression, one could also find the coefficient of  $(\zeta - \zeta_0)(\bar{\zeta} - \bar{\zeta}_0)$  in the expansion of  $K(\zeta)$  near  $\zeta_0$ , as  $K$  is real analytic away from the positive real axis.)

Since  $G$  is analytic below the horizontal line  $\text{Im } z = -1$ ,  $\partial_{\bar{\zeta}} K(\zeta) = 0$  outside of the disk of radius  $e^{2\pi}$ . Above this line, we have the estimates in Lemma 5.4-i that provides us with

$$|\partial_{\zeta \bar{\zeta}} K(\zeta)| \preceq \frac{\alpha}{r} |\zeta|^{\alpha-1}.$$

Fix  $\zeta_0 \in \mathbb{C}$  and choose a disk  $B(0, R)$  of radius  $R > e^{2\pi}$  containing  $\zeta_0$ . The general form of Cauchy Integral Formula states that for the continuous function  $\partial_\zeta K$

$$\partial_\zeta K(\zeta_0) = \frac{1}{2\pi i} \int_{\partial B(0, R)} \frac{\partial_\zeta K(\zeta)}{\zeta - \zeta_0} d\zeta + \frac{1}{2\pi i} \iint_{B(0, R)} \frac{\partial_{\bar{\zeta}} \partial_\zeta K(\zeta)}{\zeta - \zeta_0} d\zeta d\bar{\zeta}$$

Outside of the disk  $B(0, e^{2\pi})$ ,  $\partial_\zeta K = e^{2\pi i G(0,0)}$  and  $\partial_{\bar{\zeta}} \bar{\zeta} = 0$  by Lemma 5.4, thus the above formula reduces to

$$\partial_\zeta K(\zeta_0) = 1 + \frac{1}{2\pi i} \iint_{B(0, e^{2\pi})} \frac{\partial_{\bar{\zeta}} \partial_\zeta K(\zeta)}{\zeta - \zeta_0} d\zeta d\bar{\zeta}.$$

We may now estimate the difference at  $\zeta_0 \in B(0, 1)$  as

$$\begin{aligned} |\partial_{\zeta} K(\zeta_0) - \partial_{\zeta} K(0)| &\leq \frac{1}{2\pi} \iint_{B(0, e^{2\pi})} |\partial_{\bar{\zeta}} K(\zeta)| \frac{|\zeta_0|}{|\zeta - \zeta_0| \cdot |\zeta|} d\zeta d\bar{\zeta} \\ &\preceq \frac{\alpha}{r} |\zeta_0|^\alpha \iint_{B(0, e^{2\pi})} \frac{|\zeta_0|^{1-\alpha}}{|\zeta - \zeta_0| \cdot |\zeta|^{2-\alpha}} d\zeta d\bar{\zeta} \\ &\preceq \frac{1}{r} |\zeta_0|^\alpha. \end{aligned}$$

The last inequality is obtained by virtue of the following calculations.

Define  $B_1 := B(0, |\zeta_0|/2)$ ,  $B_2 := B(\zeta_0, |\zeta_0|/2)$ , and  $B_3 := B(0, e^{2\pi}) \setminus (B_1 \cup B_2)$ . Note that on  $B_1$ , we have  $|\zeta - \zeta_0| \geq |\zeta_0|/2$ , on  $B_2$  we have  $|\zeta| \geq |\zeta_0|/2$ , and on  $B_3$  we have  $|\zeta - \zeta_0| \geq |\zeta|/2$ . Hence

$$\begin{aligned} \iint_{B_1} \frac{|\zeta_0|^{1-\alpha}}{|\zeta - \zeta_0| \cdot |\zeta|^{2-\alpha}} d\zeta d\bar{\zeta} &\leq \frac{2}{|\zeta_0|^\alpha} \iint_{B_1} \frac{1}{|\zeta|^{2-\alpha}} d\zeta d\bar{\zeta} = \frac{2^{3-\alpha}\pi}{\alpha}, \\ \iint_{B_2} \frac{|\zeta_0|^{1-\alpha}}{|\zeta - \zeta_0| \cdot |\zeta|^{2-\alpha}} d\zeta d\bar{\zeta} &\leq \frac{2^{2-\alpha}}{|\zeta_0|} \iint_{B_2} \frac{1}{|\zeta - \zeta_0|} d\zeta d\bar{\zeta} = 2^{3-\alpha}\pi, \end{aligned}$$

and

$$\begin{aligned} \iint_{B_3} \frac{|\zeta_0|^{1-\alpha}}{|\zeta - \zeta_0| \cdot |\zeta|^{2-\alpha}} d\zeta d\bar{\zeta} &\leq 2|\zeta_0|^{1-\alpha} \iint_{B_3} \frac{1}{|\zeta|^{3-\alpha}} d\zeta d\bar{\zeta} \\ &\leq 2|\zeta_0|^{1-\alpha} \iint_{B(0, e^{2\pi}) \setminus B_1} \frac{1}{|\zeta|^{3-\alpha}} d\zeta d\bar{\zeta} \\ &= \frac{8\pi|\zeta_0|^{1-\alpha}}{\alpha-1} (e^{2\pi(\alpha-1)} - (|\zeta_0|/2)^{\alpha-1}) \\ &\leq \frac{2^{4-\alpha}\pi}{1-\alpha}. \end{aligned}$$

□

**Lemma 5.6.** — *We have the Inequality (18) on the upper half plane.*

*Proof.* — This results from Lemmas 5.5 and 5.4-ii, using Equations (21) and (22). □

**Remark.** — An alternative approach to get Inequality (18) from the model map  $H$  is using the Beltrami differential equation. It follows from the properties of  $H$  that the complex dilatation of  $H$ ,  $\mu := \partial_{\bar{z}} H / \partial_z H$ , can be extended to a  $C^1$  map onto the upper half plane, using the relation  $\mu(z+1) = \mu(z)$ . The function  $\mu$  has absolute value strictly less than 1 at points with large imaginary part. The Beltrami equation  $\mu \partial_z G = \partial_{\bar{z}} G$  has a  $C^2$  periodic solution with Fourier expansion. Indeed, one can find the coefficients of this expansion in terms of  $a_j$ 's and  $b_j$ 's, by comparing the coefficients in the Beltrami equation term by term.

#### 5.4. Main Estimate. —

**Proposition 5.7 (main estimate).** — *There exists a constant  $M$  such that for every  $r \in (0, 1/2]$ , and every  $w \in \text{Dom } L_h \cap \Theta(r/\alpha) \cap \Theta(C_2 + 1)$ , we have*

$$(23) \quad |L'_h(w) - 1| \leq \frac{M}{r} e^{-2\pi\alpha \text{Im } w}.$$

*Proof.* — Since  $|L'_h(w)|$  is uniformly bounded by Lemma 5.2, it is enough to prove the proposition for  $w$  with  $\text{Im } w \geq 0$ . Given such  $w$ , we choose a  $\mathcal{K}_h \subset \Theta(r/\alpha) \cap \Theta(C_2 + 1)$  containing  $w$ , and consider the corresponding map  $H$  and  $G$ . (Here  $\text{Im } A \geq 0$ .)

By Lemmas 5.3 and 5.4, at  $z := H^{-1}(w)$  we have

$$|\partial_z H(z) - 1| \leq M_1 \frac{\alpha}{r} e^{-2\pi\alpha \text{Im } w}, \text{ and } |\partial_z G(z) - 1| \leq M_2 \frac{1}{r} e^{-2\pi\alpha \text{Im } w},$$

for some constants  $M_1$  and  $M_2$  independent of  $r$  and  $w$  (and  $G, H$ ).

Applying  $\partial_z$  to  $G = L_h \circ H$  at  $w = H(z)$ , we obtain

$$\partial_z G(z) = \partial_w L_h(w) \cdot \partial_z H(z).$$

This implies the desired inequality.  $\square$

#### 5.5. Proof of the main technical lemmas. —

**Lemma 5.8.** — *For every  $h \in \mathcal{IS}_{(0, \alpha^*]}$  and  $z \in \mathbb{C}$ , let  $\mathbb{L}og$  denote an arbitrary inverse branch of  $\mathbb{E}xp$  containing  $\tau_h(z)$  in the interior of its domain. We have*

$$|(\mathbb{L}og \circ \tau_h)'(z) - \alpha| \leq C_8 \alpha e^{-2\pi\alpha \text{Im } z},$$

where  $C_8$  is a constant independent of  $h$  and  $w$ .

*Proof.* — The proof follows from explicit calculations.  $\square$

*Proof of Lemma 3.4.* — By definition,  $\chi_h := \mathbb{L}og \circ (\Phi_h^\dagger)^{-1}$ , and  $(\Phi_h^\dagger)^{-1} = \tau_h \circ L_h^{-1}$ . To estimate the derivative of  $\chi_h$ , first we estimate the derivative of the inverse map  $L_h^{-1}$  at  $w$  using Lemma 5.7. To this end, we need to locate  $L_h^{-1}(w)$ .

Given  $w \in \Theta(r, \alpha) \cap \text{Dom } \chi_h$ , by pre-compactness of the class  $\mathcal{IS}_{(0, \alpha^*]}$ , and Lemma 5.2, there exists a  $r'$ , with  $r'/r$  uniformly bounded, such that  $L_h^{-1}(w)$  belongs to  $\Theta(r'/\alpha)$ .

On the other hand, from Lemma 5.1-iv, we know that  $\text{Im } L_h^{-1}(1/2\alpha)$  is bigger than  $-C_4 \log(1 + 1/2\alpha)$ . Now, from the first part of the same lemma and the relation  $L_h(F_h(w)) = L_h(w) + 1$  one concludes that  $\text{Im } L_h^{-1}(w)$  must be at least

$$-C_4 \log(1 + 1/2\alpha) - 1/4\alpha.$$

Now choose  $\delta \in \mathbb{R}$  such that

$$-C_4 \log(1 + 1/2\alpha) - 1/4\alpha \geq \delta/\alpha$$

holds for every  $\alpha \in (0, 1)$ . The above argument implies that

$$\text{Im } L_h^{-1}(w) \geq -C_4 \log(1 + 1/2\alpha) - 1/4\alpha + \text{Im } L_h^{-1}(\mathbf{i} \text{Im } w + 1/2\alpha),$$

which is at least  $\delta/\alpha + \text{Im } w - 4(C_7 + 8)$ , by the second statement in Lemma 5.4-iii, and a similar property for  $H$ .

Now we may use Lemma 5.7, at  $w' := L_h^{-1}(w)$ , to obtain

$$|L'_h(w') - 1| \leq \frac{M'}{r} e^{-2\pi\alpha \operatorname{Im} w},$$

for an appropriate constant  $M'$ .

With  $z = L_h^{-1}(w)$ ,

$$\begin{aligned} |\chi'_h(w) - \alpha| &\leq |(\mathbb{L}og \circ \tau_h)'(z) \cdot (L_h^{-1})'(z) - (\mathbb{L}og \circ \tau_h)'(w) + (\mathbb{L}og \circ \tau_h)'(z) - \alpha| \\ &\leq |(\mathbb{L}og \circ \tau_h)'(z)| |(L_h^{-1})'(z) - 1| + |(\mathbb{L}og \circ \tau_h)'(z) - \alpha| \\ &\leq C \frac{\alpha}{r} e^{-2\pi\alpha \operatorname{Im} w}, \end{aligned}$$

for some constant  $C$ . □

*Proof of Lemma 3.9.* — We show that if for some  $m$ ,  $\operatorname{Im} \zeta_m \leq D/\alpha_m$ , then the inequality in the lemma holds for  $m-1$ . Indeed, if  $\operatorname{Im} \zeta_m \leq D/\alpha_m$ , by Lemma 5.2,

$$\begin{aligned} \operatorname{Im} L_{f_m}^{-1}(\zeta_m) &\leq C_5(D/\alpha_m + 1/\alpha_m) + \operatorname{Im} \operatorname{cv}_{F_{f_m}} \\ &\leq C_5(D+1)/\alpha_m + C_2. \end{aligned}$$

An explicit calculation on  $\mathbb{L}og \circ \tau_{f_m}$  implies that

$$\begin{aligned} \operatorname{Im} \zeta_{m-1} &= \operatorname{Im} \mathbb{L}og \circ \phi_m^{-1}(\zeta_m) \\ &= \operatorname{Im} \mathbb{L}og \circ \tau_{f_m} \circ L_{f_m}^{-1}(\zeta_m) \\ &\leq \frac{1}{2\pi} \log \frac{1}{\alpha_m} + E, \end{aligned}$$

for some universal constant  $E$ . □

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